J H McDonald

MODERN ANALYTICAL GEOMETRY

AN INTRODUCTORY ACCOUNT

OF

CERTAIN MODERN IDEAS AND METHODS

IN

PLANE ANALYTICAL GEOMETRY

BY

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PREFACE.

In the following pages I have assumed on the part of the reader as much acquaintance with the processes of Cartesian Geometry and the Differential Calculus as can be obtained from any elementary text-books in these subjects; and starting from this, I have endeavoured to give a systematic account of certain ideas and methods, a familiarity with which is tacitly assumed in higher mathematics, while no adequate means of acquiring this familiarity is provided in existing English works. Among these ideas, one of the most important is that of Correspondence, and on this, in a few of its many manifestations, I have dwelt at some length. My desire has been to refrain from encroaching on what properly belongs to the theory of Higher Plane Curves—a theory so extensive, and, as is now acknowledged, so much less simple than it appeared some few years ago, that an introductory study of its fundamental conceptions may well be undertaken as a preliminary.

To a certain small extent the field here marked out coincides with that already occupied by the later chapters of Salmon's Conic Sections. Recognizing that every English-speaking student of mathematics must of necessity acquire an intimate knowledge of Dr. Salmon's incomparable treatises, I have gladly refrained from any discussion of this part, and have simply referred the reader to those chapters, adding occasionally the few words of explanation that seemed necessitated by the different order of treatment here adopted.

It has not been my ambition to add another to the many excellent collections of problems already existing, but I trust the examples scattered through the pages will be found sufficient for purposes of illustration. As these, (many of which contain results of independent importance,) are placed in general immediately after the account of the theorems on which they depend, their position sufficiently indicates the process of solution, and I have therefore included a number of them in the index.

Regarding this work as strictly introductory, I have preferred not to give too many references. Those that do appear have been given, some because they are perhaps not just in the line of reading that is usually followed, some because of special felicity of statement, a few for their historical interest. Thus the frequency of reference to any one author is not to be interpreted as an attempt to indicate the extent of my indebtedness. Had the references been so adjusted, it would have been alike my duty and my pleasure to write on every page the name of Professor Cayley.

My hearty thanks are due to various friends; to Miss I. Maddison and Miss H. S. Pearson, for help in seeing the book through the press; to Mr. F. Morley, for valuable suggestions while the work was in progress; and to Mr. J. Harkness, for his great kindness in reading the whole, not only in proof, but also in manuscript.

C. A. SCOTT.

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CONTENTS.

CHAPTER I.

POINT AND LINE COORDINATES.

CHAPTER II.

INFINITY. TRANSFORMATION OF COORDINATES.

CHAPTER III.

FIGURES DETERMINED BY FOUR ELEMENTS.

CHAPTER IV.

THE PRINCIPLE OF DUALITY.

CHAPTER V.

DESCRIPTIVE PROPERTIES OF CURVES.

CHAPTER VI.

METRIC PROPERTIES OF CURVES; THE LINE INFINITY.

CHAPTER VII.

METRIC PROPERTIES OF CURVES; THE CIRCULAR POINTS.

CHAPTER VIII.

UNICURSAL CURVES. TRACING OF CURVES.

CHAPTER IX.

CROSS-RATIO, HOMOGRAPHY, AND INVOLUTION.

CHAPTER X.

PROJECTION AND LINEAR TRANSFORMATION.

CHAPTER XI.

THEORY OF CORRESPONDENCE.

CHAPTER XII.

THE ABSOLUTE.

CHAPTER XIII.

INVARIANTS AND COVARIANTS.

CHAPTER I.

POINT AND LINE COORDINATES.

Introductory.

1. In analytical geometry the subject-matter is geometry while the language is algebraic. For progress and pleasure it is of primary importance that the language be properly adjusted to the subject; elasticity must be preserved and unnecessary restrictions cast aside. We begin therefore by examining our conceptions in analytical geometry, recognizing natural limitations, but rejecting artificial limitations except in so far as these can be shown to serve some good purpose. We begin, that is, by generalizing our conceptions and their expression as much as possible.

General Idea of Coordinates.

2. The whole of analytical geometry as hitherto studied depends on the possibility of representing the position of a point in a plane by two coordinates, with the dependent possibility of representing the position of a point in ordinary space by three coordinates. These coordinates in the case of plane geometry were regarded initially as the distances from the point to two selected lines, these distances being measured in assigned directions, viz., parallel to the selected lines. But other systems were occasionally used; for example, polar coordinates, where the distance from a fixed pole to the point, and the direction of the line joining the fixed pole to the point, were the two determining quantities; dipolar coordinates, where the two coordinates of the point were its distances from two fixed poles; the system of coordinates arranged by means of confocal conics; etc. The fundamental idea of coordinates derived from plane geometry is therefore that they are any two quantities that serve to determine the position of a point in a plane. Here there are implied certain limitations, which may be accidental; for

there are geometrical elements other than a point, whose position we may wish to specify; and we have no assurance that the number of coordinates required is necessarily two. We generalize therefore by dropping these limitations; and we say:—

Coordinates are quantities that determine the position of a geometrical element.

The nature of these quantities will depend on (1) the space assumed, (2) the problem considered, (3) the element selected.

3. We here recognize that the primary element may possibly not be a point. The point certainly presents itself naturally to our minds as *the element*, par excellence, probably because all our drawing is done with a point. But the straight line is essentially as simple; and it is possible to imagine that we might have learnt to do all our drawing with a straight-edge instead of a point. We should then regard a point as a secondary element, uniquely determined by two straight lines; and this secondary element, the point, would suggest to us an infinity of straight lines passing through it, just as with our present ideas the secondary element, the straight line, uniquely determined by two points, suggests to us an infinity of points lying on it. We shall constantly have occasion to notice in detail the correspondence between the two geometrical theories; the two that is in which, the field being restricted to the plane, the *primary elements* are respectively the point and straight line, the *secondary elements* the straight line and point.

The element, then, need not be a point; it may be some other geometrical entity.

4. In the next place we consider how many coordinates are necessary.

Our primary element is regarded as having position in space; we consider it therefore as able to change its position in certain ways, reducible to a certain number of independent ways; or we may say, the element has a certain number of *degrees of freedom*.

Suppose, for definiteness, we take for element a point, and for space a line, straight or curved. There is only one possible way in which the point can move, viz., along the line; it may of course move forward or backward, but these differ simply as positive and negative, that is they differ only in sense. The point has therefore only one degree of freedom; its position on the line is determined by one coordinate, *e.g.* its distance from some fixed point

on the line; its freedom consists in the possibility of varying this one coordinate; if the value of this one coordinate were given, the one degree of freedom would be destroyed.

Now imagine the point to be at P on any surface, for example a plane, and free to move to any other position Q on the plane. For this it is not essential that the point

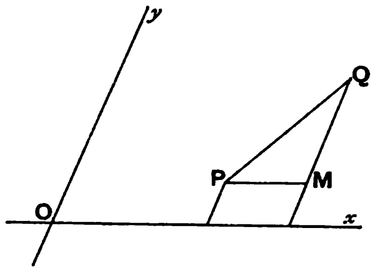

FIG. 1.

be able to move directly from P to Q; the step can be accomplished by steps in two selected directions; for example, by PM, MQ parallel to Ox and Oy (Fig. 1); thus two degrees of freedom allow for all conceivable motions of the point on the plane. These two degrees of freedom can be algebraically expressed as the variations of two coordinates, x and y; and if the value of either of these coordinates were given, one degree of freedom would be destroyed; if both coordinates were given, that is, if the position of the point were given, both degrees of freedom would be destroyed.

Similarly a point in ordinary space has three degrees of freedom, and to destroy the three, that is, to fix the position of the point, three independent coordinates must be given.

5. This same idea may be differently expressed. A point (an element) with one degree of freedom can assume a singly infinite number of positions; its one coordinate is susceptible of numerical values ranging from $-\infty$ through 0 to $+\infty$. A point (an element) with two degrees of freedom can assume a doubly infinite number of positions (∞^2); each of its two coordinates is susceptible of numerical values ranging from $-\infty$ through 0 to $+\infty$; and in general, an element with a degrees of freedom, that is, with a independent coordinates, can assume ∞^a positions.

6. These fundamental conceptions are expressed in various ways; we speak of space of one, two, three, ..., a

dimensions; of a one, two, three, or a-way spread. But it must be kept in mind that the number of dimensions of any assumed space depends on the selected element. Thus, for example, the point being the element, a plane is space of two dimensions; a circle with a fixed centre being the element, the plane is space of one dimension.

In ordinary space, planes through a line are singly infinite in number; that is, the line is to be regarded as one-dimensional, when the plane is the element; and similarly the line is one-dimensional when the point is the element. Ordinary space itself is three-dimensional, when the point is element; it is likewise three-dimensional when the plane is element. Thus we detect a correspondence between figures determined by points and figures determined by planes. The points in a plane form a two-fold infinity; the planes through a point form a two-fold infinity. Thus the point being element, the plane is a two-dimensional space contained in the three-dimensional space; and the plane being element, the point is a two-dimensional space contained in the three-dimensional space. Now two points in space determine a line; two planes likewise determine a line; thus the line takes the same part in the two theories.

Ordinary space is seen to be four-dimensional in lines; this appears from the fact that every line can be obtained by joining every point in one plane to every point in another plane.

Thus in Solid Geometry we have assemblages of points, planes, and lines; and we have corresponding assemblages of planes, points, and lines.

Ex. Considering the five regular solids as determined (1) by their vertices, (2) by their faces, show that one corresponds to itself, and that the others are in corresponding pairs.

But now confining ourselves to Plane Geometry, we can no longer regard the plane as element; we have only points and lines. The plane is two-dimensional as regards its points; it is also two-dimensional as regards its lines; for the position of a straight line depends on two independent quantities, for example, the intercepts made on the axes; or again, the number of straight lines in a plane is doubly infinite, for we obtain all straight lines by joining every point on one line to every point on another line. Thus the plane is of the same nature whether the point or line be regarded as element.

Further, as regards aggregates contained in the plane; the point being the element, the straight line is one-

dimensional; and the straight line being the element, the point is one-dimensional. This last appears from various considerations; a line through a fixed point has one degree of freedom, for it can rotate; one coordinate, for example, the angle made with a fixed direction, serves to determine the position of the line; the number of straight lines through a point is singly infinite.

We have thus one more step in tracing the correspondence already referred to (§ 3) between two special theories in plane geometry,—the two in which the point and the straight line are respectively taken as primary element.

Note. For a discussion of the different aggregates of elements in any assumed space, see Reye, *Die Geometrie der Lage*, Ch. I., or Section II. of Professor Henrici's Article on Geometry in the *Encyclopædia Britannica.*

7. We have seen that the number of independent coordinates required depends on the space and on the element. But in many cases it is convenient to use more than this necessary number of coordinates, connected by identical relations. Thus for instance we sometimes use Cartesians and polar coordinates in one piece of work, that is, four coordinates with two identical relations

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Homogeneous Point Coordinates.

8. We shall now confine the work, for the present, to geometry in a plane, using for element sometimes the point and sometimes the line. In either case the position of the element is determined by two independent coordinates; but we are at liberty to use more than two coordinates connected by relations.

In Cartesian geometry, we obtain the two coordinates of a point by means of two fixed lines; we here begin by assuming an undetermined number of fixed lines, a, b, c, etc. Let the distances from a point P to these lines be denoted by α, β, γ, etc.; using the ordinary convention of signs, each line has a positive side and a negative side, which may be initially arbitrarily assigned. Any one of these distances, *e.g.* α, determines P as lying on a line parallel to the line a; the ratio of any two, $\alpha : \beta$, determines P as lying on a line through the intersection ab. The position of P is therefore determined uniquely by two distances, α and β, or by two ratios $\alpha : \beta$, $\alpha : \gamma$, by which a third $\beta : \gamma$ is implied. If then we select three lines, a, b, c, not concurrent, the position

of P is determined by any two of the ratios $\alpha:\beta:\gamma$. It is essential that the three lines a, b, c (Fig. 2) be not concurrent, for otherwise the two lines $P(ab)$, $P(ac)$, whose intersection

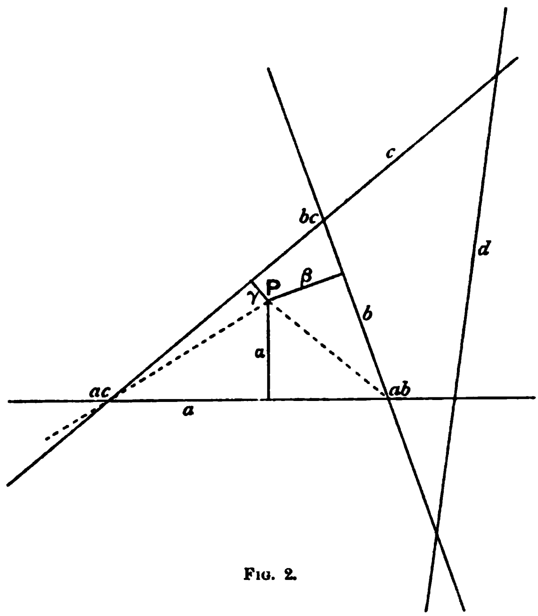

FIG. 2.

has to give P, would be identical (Fig. 3). Let the lengths of the sides of the triangle be denoted by a, b, c, and its area by Δ; and for definiteness let the positive and negative sides of

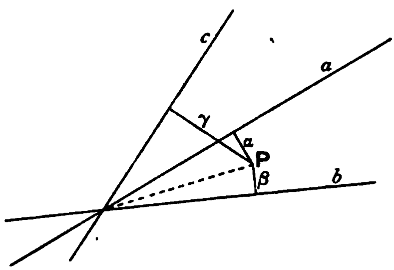

FIG. 3.

the three lines be assigned so that a point inside the triangle is on the positive side of every one of these lines. We have then

$$a\alpha+b\beta+c\gamma=2\Delta.$$

For denoting the vertices of the triangle by A, B, C, we

have for a point inside the triangle a relation among the areas, viz.

$$2\,.\,PBC+2\,.\,PCA+2\,.\,PAB=2\,.\,ABC,$$

that is, $$BC\,.\,PD+CA\,.\,PE+AB\,.\,PF=2\Delta,$$

that is, $$a\alpha+b\beta+c\gamma=2\Delta;$$

and for a point outside the triangle, *e.g.* P',

$$2\,.\,P'BC-2\,.\,P'CA+2\,.\,P'AB=2\,.\,ABC,$$

that is, $$BC\,.\,P'D'-CA\,.\,P'E'+AB\,.\,P'F'=2\Delta,$$

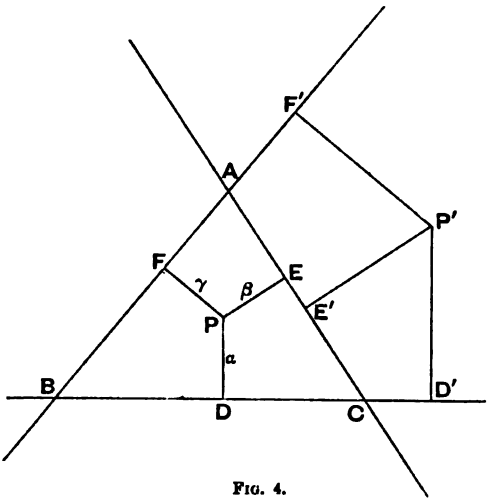

FIG. 4.

that is, $a\alpha+b\beta+c\gamma=2\Delta$, as before, since β is now (Fig. 4) represented by $-P'E'$.

Thus we have in general

$$a\alpha+b\beta+c\gamma=2\Delta,$$

and consequently when the ratios $\alpha:\beta:\gamma$ are given, the values of α, β, γ are known.

Now one advantage of using the ratios is that all our equations can be made homogeneous in α, β, γ. For the two independent coordinates of P being $\alpha:\gamma$, $\beta:\gamma$, any fact about the position of P is expressed by an equation $F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}\right)=0$, of degree n; multiplying throughout by γ^n, this becomes homogeneous. Or again, if α, β be taken as coordinates, then by means of the relation $a\alpha+b\beta+c\gamma=2\Delta$, which can be written in the form $\frac{a}{2\Delta}\alpha+\frac{b}{2\Delta}\beta+\frac{c}{2\Delta}\gamma=1$, any non-homogeneous expression can be made homogeneous; for the terms of lower

degree can be multiplied by any desired power of the unit multiplier

$$\frac{a}{2\Delta}\alpha+\frac{b}{2\Delta}\beta+\frac{c}{2\Delta}\gamma.$$

9. From our definitions it follows that for any point on the line a we have $\alpha=0$; that is, the line a is denoted by $\alpha=0$; thus the equations of the three fundamental lines are

$$\alpha=0, \quad \beta=0, \quad \gamma=0.$$

Now consider any fourth line $\delta=0$. By means of the triangle abc we proved that α, β, γ must satisfy a relation which may be written in the form

$$a'\alpha+b'\beta+c'\gamma=1.$$

Similarly by means of the triangle abd we find that α, β, δ must satisfy a relation

$$a''\alpha+b''\beta+d''\delta=1.$$

Subtracting, we see that α, β, γ, δ must satisfy a relation

$$(a'-a'')\alpha+(b'-b'')\beta+c'\gamma-d''\delta=0,$$

whence $$\delta=\frac{a'-a''}{d''}\alpha+\frac{b'-b''}{d''}\beta+\frac{c'}{d''}\gamma,$$

that is, $$\delta=f\alpha+g\beta+h\gamma.$$

Note. If the line $\delta=0$ pass through the intersection of the lines $\alpha=0$, $\beta=0$, there is not any triangle abd; but there is a triangle acd.

Thus any fourth distance, δ, is a linear function of α, β, γ; and the equation of any fourth line, $\delta=0$, when expressed in terms of α, β, γ, becomes

$$f\alpha+g\beta+h\gamma=0;$$

that is, in this system of coordinates *any straight line is represented by a linear equation.*

10. There is no occasion to limit ourselves unduly in this choice of coordinates; the position of P will be equally determined if instead of $\alpha:\beta:\gamma$ we know $l\alpha:m\beta:n\gamma$, where l, m, n are any multipliers, initially arbitrarily chosen. We write therefore $x, y, z=l\alpha, m\beta, n\gamma$; by this change the fundamental identical relation $a\alpha+b\beta+c\gamma=2\Delta$ becomes

$$\frac{a}{l}x+\frac{b}{m}y+\frac{c}{n}z=2\Delta,$$

that is, $$\frac{a}{2\Delta l}x+\frac{b}{2\Delta m}y+\frac{c}{2\Delta n}z=1,$$

which may be written

$$a_0x+b_0y+c_0z=1;$$

and the equation of any line

$$f\alpha+g\beta+h\gamma=0$$

becomes

$$f'x+g'y+h'z=0,$$

that is, it remains a homogeneous linear equation.

Recalling to mind the significance of δ, we see that the distance from any point x, y, z to a line $fx+gy+hz=0$ is a multiple of $fx+gy+hz$; that is, *the distance from any point x, y, z to a line $u=0$ is a multiple of the corresponding value of the expression u.*

11. It may be convenient occasionally to retain more than three coordinates, with homogeneous identical relations; that is, any number of coordinates δ may be kept, with relations $\delta\equiv l\alpha+m\beta+n\gamma$. Thus, for example, if the line $x+y+z=0$ be an important one in a problem, it may be convenient to write $-u$ for $x+y+z$, so that we have four coordinates x, y, z, u, connected by the identity $x+y+z+u\equiv 0$.

Suppose we have two lines $u=0$, $v=0$, where u, v, can when we please be written as linear functions of x, y, z. Then $u+kv=0$ is a line through the intersection of $u=0$, $v=0$. For the values of x, y, z, that make the linear expressions u, v, vanish, make $u+kv$ vanish; that is, the point that lies on each of the lines $u=0$, $v=0$, is a point on $u+kv=0$.

Note. Letters such as u, v, etc., will be used not only as abbreviations for linear (or other) expressions, but also for referring to the diagram. Thus the line u means the line marked with a letter u in the diagram; and when the equation of this line is written in abridged form, it will be written $u=0$. The point uv is the intersection of the lines $u=0$, $v=0$; *e.g.* in the fundamental triangle A, B, C are respectively yz, zx, xy.

Homogeneous Line Coordinates.

12. Now regarding the line as element, let us assign its position in the corresponding way. The position of the point was referred to fixed fundamental lines; we refer the position of the line to fixed fundamental points. Let the distances to the line from these fixed points A, B, C, etc. be denoted by p, q, r, etc., so that the statement $p=0$ means that the line passes through A, and so on. Regarding the line, as usual, as having a positive and a negative side, we see that any two distances, q, r, must be considered as being of the same sign when B, C, are on the same side of the line.

The absolute values of two distances, q, r, determine the line as one of the four common tangents to two circles whose centres are B, C, and whose radii are q, r; a com-

parison of the signs of q, r shows that the line is one of two; in Fig. 5 q, r have the same sign for 1, 2, and opposite signs for 3, 4. Thus in general the values of two distances q, r, determine the line as one of two.

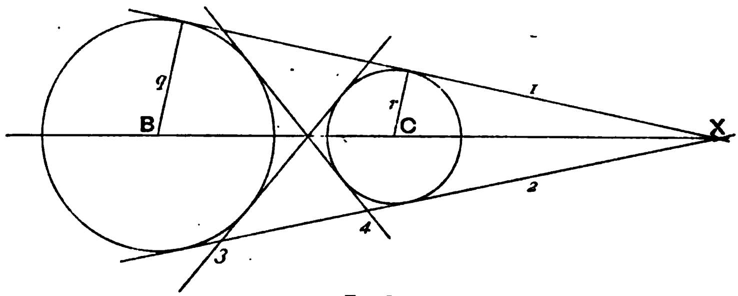

FIG. 5.

But if we deal with ratios of distances, the position of the line is determined by means of the points in which it meets BC, CA, AB (the three fundamental points A, B, C, having been chosen non-collinear), just as when we are dealing with ratios of distances from a point P to three non-concurrent fundamental lines the position of P is determined by means of the lines joining it to the points bc, ca, ab. For if the line meet BC in X, we have (Fig. 5) by similar triangles,

$$BX : CX = q : r,$$

and the position of X is uniquely determined by the ratio $q : r$. Hence two such ratios, $q : r$, $r : p$ (which imply the third, $p : q$) determine two points X, Y on the line, and thus they determine the line itself. That is, the line is determined by the ratios $p : q : r$.

13. Let us now compare the determination of the position of the line by means of p, q, r, with the results obtained in the system of homogeneous point coordinates already discussed. In order to keep the connection between the two theories as clear as possible, and the transition from one to the other as simple as possible, we take the points A, B, C, determined by the triangle of reference in the system of point coordinates, for fundamental points in the system of line coordinates now under investigation. Let the line referred to the system of point coordinates have the equation

$$f\alpha + g\beta + h\gamma = 0;$$

the line is therefore determined by f, g, h; it is equally determined by p, q, r; consequently these two sets of quantities must be expressible in terms of one another.

Let X (Fig. 6) be $\alpha_1, \beta_1, \gamma_1$, then $\alpha_1=0$, and therefore from the assigned equation of the line XYZ, $g\beta_1+h\gamma_1=0$. From the figure, X being on the negative side of CA and on the positive side of AB,

$$q:r=BX:CX=\text{area } BXA:\text{area } CXA=c\gamma_1:-b\beta_1,$$

therefore $$bq:cr=\gamma_1:-\beta_1=g:h;$$

and similarly $$ap:bq=f:g;$$

hence $$f:g:h=ap:bq:cr.$$

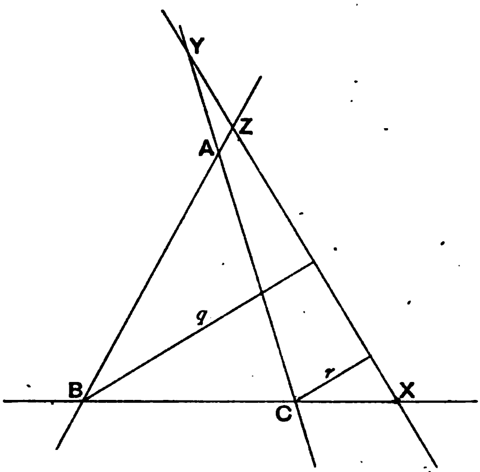

FIG. 6.

Thus the coefficients in the equation of the line in the system of point coordinates are linearly expressible in terms of the coordinates of the line in the system of line coordinates.

14. This same result may be expressed in a form slightly different and more significant. Let α, β, γ be any point on the line, then $f\alpha+g\beta+h\gamma=0$. But regarding the line as determined by p, q, r, we write this condition by what has just been proved in the form $ap\alpha+bq\beta+cr\gamma=0$; that is, the point α, β, γ and the line p, q, r are *united in position* if the condition $ap\alpha+bq\beta+cr\gamma=0$ be satisfied.

15. We have already noticed that we are at liberty to take for coordinates of the point any multiples we please of α, β, γ, viz., $x, y, z=l\alpha, m\beta, n\gamma$. Similarly for coordinates of the line we are at liberty to take any multiples we please of p, q, r, viz., $\xi, \eta, \zeta=\lambda p, \mu q, \nu r$. Making these changes, the

relation just found becomes:—The point x, y, z and the line ξ, η, ζ are united in position if the condition

$$\frac{a}{l\lambda}x\xi+\frac{b}{m\mu}y\eta+\frac{c}{n\nu}z\zeta=0$$

be satisfied. Since we constantly use both systems of coordinates in one investigation, there is a decided practical convenience in adjusting the two systems of multipliers l, m, n and λ, μ, ν so that this condition may reduce to the simplest possible form; we therefore choose these quantities so that

$$\frac{a}{l\lambda}=\frac{b}{m\mu}=\frac{c}{n\nu};$$

that is, we take $\qquad l\lambda:m\mu:n\nu=a:b:c;$

and the relation becomes:—

The point x, y, z and the line ξ, η, ζ are united in position if

$$x\xi+y\eta+z\zeta=0.$$

Relation of the Two Systems.

16. The equations $l\lambda:m\mu:n\nu=a:b:c$ leave us free to choose either set of ratios $l:m:n$ or $\lambda:\mu:\nu$, but they then determine the other set. We may therefore choose our system of point coordinates, but then the corresponding system of line coordinates is at once deduced.

As particular systems of point coordinates we have

I. *Trilinears.* Here x, y, z are simply proportional to α, β, γ, *i.e.* $\qquad x:y:z=\alpha:\beta:\gamma,$

therefore $\qquad l=m=n.$

In this system the fundamental identical relation is

$$ax+by+cz=\text{constant}.$$

The associated system of line coordinates is determined by

$$l\lambda:m\mu:n\nu=a:b:c,$$

that is, by $\qquad \lambda:\mu:\nu=a:b:c,$

and thus $\qquad \xi:\eta:\zeta=ap:bq:cr.$

II. *Areals*, also called triangular coordinates. In this system the coordinates of a point P are the ratios of the triangles PBC, PCA, PAB to the whole triangle ABC. Now 2 area $PBC=a\alpha$, therefore

$$x=\frac{a\alpha}{2\Delta},\quad y=\frac{b\beta}{2\Delta},\quad z=\frac{c\gamma}{2\Delta},$$

that is, $\qquad l, m, n=\dfrac{a}{2\Delta}, \dfrac{b}{2\Delta}, \dfrac{c}{2\Delta}.$

The fundamental identical relation is now

$$x+y+z=1.$$

The associated system of line coordinates is simply

$$\xi : \eta : \zeta = p : q : r,$$

for the equations giving λ, μ, ν reduce to

$$\lambda=\mu=\nu.$$

17. The condition $x\xi+y\eta+z\zeta=0$, expressing the union of the point x, y, z and the line ξ, η, ζ, is of primary importance. Writing f, g, h for ξ, η, ζ it may be stated as follows:—

If the equation of a line be $fx+gy+hz=0$, the coordinates of the line are f, g, h.

It thus enables us to pass from the expression of a line in point coordinates to its expression in line coordinates.

But again it may be read differently:—The line whose coordinates are ξ, η, ζ passes through the point x, y, z if

$$x\xi+y\eta+z\zeta=0;$$

that is, any line with coordinates ξ, η, ζ passes through the point f, g, h if $f\xi+g\eta+h\zeta=0$. Thus in order that a variable line may pass through a fixed point the coordinates of the line must satisfy an equation of the first degree. This equation is called the equation of the point; it is the equation that must be satisfied by the coordinates of all primary elements that are united with the assigned secondary element; it is therefore exactly analogous to the equation of a line in point coordinates, for this is simply the equation that must be satisfied by the coordinates of all points that lie on the assigned line.

If then f, g, h be the point coordinates of a point, its equation in line coordinates is $f\xi+g\eta+h\zeta=0$.

The distance from a point l, m, n to a line $\xi x+\eta y+\zeta z=0$ was shown in § 10 to be a multiple of $\xi l+\eta m+\zeta n$; hence the distance from a point $l\xi+m\eta+n\zeta=0$ to a line ξ, η, ζ is a multiple of $l\xi+m\eta+n\zeta$; that is, *the distance to any line ξ, η, ζ from a point $\varpi=0$ is a multiple of the corresponding value of the expression ϖ.*

18. Just as in the case of point coordinates, we are at liberty to use more than three fundamental points; let any additional one be $\theta=0$, then θ can at any stage of the work be written as a linear function of ξ, η, ζ.

Suppose we have any two points whose equations are $\varpi=0$, $\rho=0$, where ϖ, ρ are linear functions of ξ, η, ζ. Then $\varpi+k\rho=0$ being linear is the equation of some point; now coordinates ξ, η, ζ that make the linear expressions ϖ, ρ vanish

simultaneously, make $\varpi+k\rho$ vanish; thus the line that passes through each of the points $\varpi=0$, $\rho=0$, passes through the point $\varpi+k\rho=0$; *i.e.* $\varpi+k\rho=0$ is a point on the line $\varpi\rho$. (See § 11.)

Note. Letters such as θ, ρ, etc., will be used not only as abbreviations for linear (or other) expressions in ξ, η, ζ, but also for referring to the diagram. Thus the point ρ means the point marked with a letter ρ in the diagram; and when the equation of this point is written in abridged form, it will be written $\rho=0$; the line $\varpi\rho$ is the join of the points $\varpi=0$, $\rho=0$; *e.g.* in the fundamental triangle, A, B, C are the points ξ, η, ζ, their equations being $\xi=0$, $\eta=0$, $\zeta=0$; a, b, c are the lines $\eta\zeta$, $\zeta\xi$, $\xi\eta$. Ordinary Roman capitals will however frequently be used simply to denote point-positions in the diagrams.

19. Here we have an important step in the correspondence already noted between the two geometries in a plane, the two, that is, in which the primary element is taken to be (1) the point, (2) the straight line. These two theories now run as follows:—

We may regard the *point* as element; two points determine a *line*; an indefinite number of points lie on the line.

The position of a point in the plane is determined by two independent coordinates; but it is convenient to make use of three coordinates.

By the *equation of a line* we mean the relation satisfied by the coordinates of all points on the line; this is of the first degree

$$fx+gy+hz=0.$$

The equation of the line determined by the points (x_1, y_1, z_1), (x_2, y_2, z_2) is

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=0.$$

We may regard the *line* as element; two lines determine a *point*; an indefinite number of lines pass through the point.

The position of a line in the plane is determined by two independent coordinates; but it is convenient to make use of three coordinates.

By the *equation of a point* we mean the relation satisfied by the coordinates of all lines through the point; this is of the first degree

$$f\xi+g\eta+h\zeta=0.$$

The equation of the point determined by the lines (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) is

$$\begin{vmatrix} \xi & \eta & \zeta \\ \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \end{vmatrix}=0.$$

This is obtained by considering the linear equation $fx+gy+hz=0$, which has to be satisfied by two sets of quantities (x_1, y_1, z_1), (x_2, y_2, z_2).

The condition that three points 1, 2, 3 be collinear is the vanishing of the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

The condition that three lines 1, 2, 3 be concurrent is the vanishing of the determinant

$$\begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ \xi_3 & \eta_3 & \zeta_3 \end{vmatrix}.$$

For the three linear equations $fx_1+gy_1+hz_1=0$, $fx_2+gy_2+hz_2=0$, $fx_3+gy_3+hz_3=0$, must be satisfied; eliminating f, g, h, the result follows.

The coordinates of the intersection of the two lines

$$fx+gy+hz=0,$$
$$f'x+g'y+h'z=0,$$

are $gh'-g'h,\ hf'-h'f,\ fg'-f'g$.

The coordinates of the join of the two points

$$f\xi+g\eta+h\zeta=0,$$
$$f'\xi+g'\eta+h'\zeta=0,$$

are $gh'-g'h,\ hf'-h'f,\ fg'-f'g$.

These are obtained by solving the equations.

The condition that three lines

$$f_1x+g_1y+h_1z=0,$$
$$f_2x+g_2y+h_2z=0,$$
$$f_3x+g_3y+h_3z=0,$$

be concurrent is the vanishing of the determinant

$$\begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}.$$

The condition that three points

$$f_1\xi+g_1\eta+h_1\zeta=0,$$
$$f_2\xi+g_2\eta+h_2\zeta=0,$$
$$f_3\xi+g_3\eta+h_3\zeta=0,$$

be collinear is the vanishing of the determinant

$$\begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix}.$$

This is obtained by eliminating the variables from the given equations.

The coordinates of any point on the join of (x_1, y_1, z_1), (x_2, y_2, z_2) are expressible in the form

$$lx_1+mx_2,\quad ly_1+my_2,\quad lz_1+mz_2.$$

The coordinates of any line through the intersection of (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are expressible in the form

$$l\xi_1+m\xi_2,\quad l\eta_1+m\eta_2,\quad l\zeta_1+m\zeta_2.$$

For (x, y, z), (x_1, y_1, z_1), (x_2, y_2, z_2) are collinear if, and only if

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=0.$$

Hence determining l, m, so that

$$ly_1+my_2=y,\quad lz_1+mz_2=z,$$

the determinant shows that

$$\begin{vmatrix} x-lx_1-mx_2 & 0 & 0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}=0,$$

that is, unless $y_1z_2-y_2z_1=0$, $\quad x=lx_1+mx_2$.

This possible interference, $y_1z_2-y_2z_1=0$, would however be noticed earlier, for it would prevent the determination of l, m as directed. But if $y_1z_2-y_2z_1=0$, certainly $x_1z_2-x_2z_1\neq0$, for otherwise the ratios $x_1:y_1:z_1$ would be the same as $x_2:y_2:z_2$, *i.e.* the points 1, 2 would be the same. Thus in this case we determine l, m from the equations

$$lx_1+mx_2=x,\quad lz_1+mz_2=z.$$

The underlying principle manifested in this correspondence is known as the Principle of Duality. The meaning of the name, the importance of the principle, and the utility of the correspondence, will appear more plainly in the following chapters.

20. We have found that the actual distances from a point to the sides of the triangle of reference satisfy a permanent relation, viz., $a\alpha+b\beta+c\gamma=2\Delta$; and that hence the actual values of the coordinates in any homogeneous point system satisfy a permanent relation of the form

$$a_0x+b_0y+c_0z=1.$$

Similarly the actual distances to a line from the vertices of the triangle of reference satisfy a permanent relation, viz.,

$$a^2p^2+b^2q^2+c^2r^2-2bq\,.\,cr\,.\cos A$$
$$-2cr\,.\,ap\,.\cos B-2ap\,.\,bq\,.\cos C=4\Delta^2.$$

Let the line make with BC, CA, AB, angles θ, ϕ, ψ (Fig. 7), so that

$$\psi-\phi=A,\quad \theta-\psi=B,\quad \theta-\phi=\pi-C.$$

Considering the quadrilaterals $BCFE$ etc., we have

$$\text{area } ABC=BCFE-CFDA+ABED,$$

therefore $\quad 2\Delta=(q+r)FE-(r+p)FD+(p+q)ED$

$$=-p\,.\,FE+q\,.\,FD-r\,.\,ED.$$

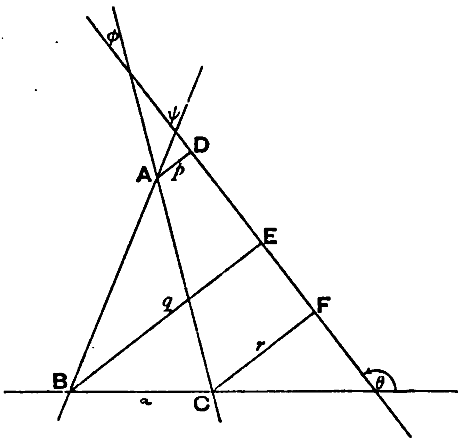

FIG. 7.

Now $\quad FE=a\cos(\pi-\theta),\quad FD=b\cos\phi,\quad ED=c\cos\psi,$

therefore $\quad ap\cos\theta+bq\cos\phi-cr\cos\psi=2\Delta$..............(1).

Also, from the identity

$$p(q-r)+q(r-p)+r(p-q)=0,$$

by means of the relations

$$q-r=a\sin\theta,\quad r-p=b\sin\phi,\quad p-q=-c\sin\psi,$$

we obtain $\quad ap\sin\theta+bq\sin\phi-cr\sin\psi=0$................(2).

From (1) and (2), by squaring and adding, we find

$$a^2p^2+b^2q^2+c^2r^2-2bq\,.\,cr\,(\cos\phi\cos\psi+\sin\phi\sin\psi)$$
$$-2\,cr\,.\,ap\,(\cos\theta\cos\psi+\sin\theta\sin\psi)$$
$$+2ap\,.\,bq\,(\cos\theta\cos\phi+\sin\theta\sin\phi)=4\Delta^2.$$

The expressions within brackets being

$$\cos(\psi-\phi),\quad \cos(\theta-\psi),\quad \text{and}\quad \cos(\theta-\phi),$$

that is, $\cos A$, $\cos B$, and $-\cos C$,

this relation reduces to

$$a^2p^2+b^2q^2+c^2r^2-2bq\,.\,cr\,.\cos A-2cr\,.\,ap\,.\cos B$$
$$-2ap\,.\,bq\,.\cos C=4\Delta^2.$$

The fundamental identical relation is therefore of a different form in the two systems considered; that is, the correspondence hitherto noted between the two theories does not appear to hold. This will be investigated later (Ch. VII. and XII.); for the present we shall not use the Principle of Duality in work that depends on the fundamental identical relation, that is, in work that requires us to use actual values of the coordinates; we shall not use it in investigating *metric* relations.

When the point coordinates are trilinears, we have agreed to use as the associated line system $\xi, \eta, \zeta = ap, bq, cr$. With these coordinates the relation just found becomes

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C=4\Delta^2.$$

Similarly for any other system of coordinates we have a modified form of each of the two fundamental relations, which can be deduced at once from the natural forms

$$a\alpha+b\beta+c\gamma=2\Delta,$$
$$a^2p^2+b^2q^2+c^2r^2-2bq\,.\,cr\cos A$$
$$-2cr\,.\,ap\,.\cos B-2ap\,.\,bq\,.\cos C=4\Delta^2.$$

Distance from a Point to a Line.

21. General homogeneous coordinates do not lend themselves readily to the direct investigation of purely metric properties. Formulæ on which these investigations depend (*e.g.* for the distance from one point to another, for the angle made by one line with another, etc.) can of course be obtained, and may be found, *e.g.* in Ferrers' *Trilinear Coordinates.* But the principle here adopted of using only suitable methods for any assigned problem, and applying given methods only to suitable problems, forbids their introduction. One particular symmetrical metric expression must however be found, viz., that for the distance from a point to a line. This is of importance in

the discussion of the metric relations involved in the Principle of Duality (Ch. II., VII., and XII.), and moreover it relates to the conception that has been made the foundation of our double system of coordinates.

Let OM be the distance from O, the given point α, β, γ, to the given line p, q, r, this given line making with CB the angle θ. Draw from A, O, lines parallel and perpendicular to BC, meeting in E, and join OB (Fig. 8). Then we have (indicating signs of magnitudes by signs prefixed, and not by the order of the letters),

$$OM = q - OB \sin(\theta + DBO),$$

that is $$OM = q - OB \cos(\theta - BOD);$$

also $$OM = r - OC \cos(\theta - COD),$$

and $$OM = p - OA \sin\left(AOM - \frac{\pi}{2}\right).$$

Now $$AOM + AOD = \pi + \theta,$$

therefore $$AOM - \frac{\pi}{2} = \frac{\pi}{2} + \theta - AOD;$$

hence $$OM = p - OA \cos(\theta - AOD).$$

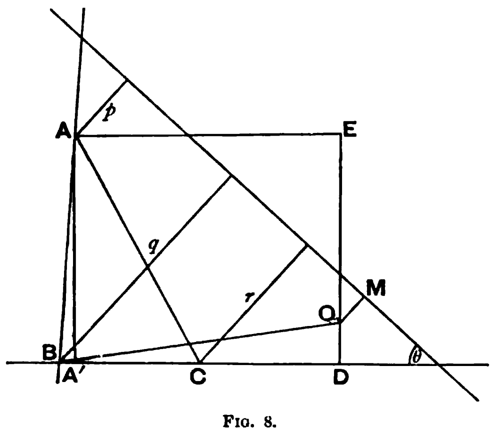

FIG. 8.

From the three expressions here found for OM we obtain

$$\begin{aligned} p - OM &= -OE \,.\cos\theta + AE \,.\sin\theta, \\ q - OM &= OD \,.\cos\theta + BD \,.\sin\theta, \\ r - OM &= OD \,.\cos\theta + CD \,.\sin\theta; \end{aligned}$$

whence, multiplying by $a\alpha$, $b\beta$, $c\gamma$, and adding,

$$\begin{aligned} a\alpha p + b\beta q + c\gamma r - OM(a\alpha + b\beta + c\gamma) \\ = \cos\theta(-OE \,.\, a\alpha + OD \,.\, b\beta + OD \,.\, c\gamma) \\ + \sin\theta(AE \,.\, a\alpha + BD \,.\, b\beta + CD \,.\, c\gamma). \end{aligned}$$

In this the coefficient of $\cos\theta$ is

$$OD(a\alpha+b\beta+c\gamma)-(OE+OD)a\alpha,$$
$$=OD\,.\,2\Delta-AA'.\,a\alpha, \quad =2\Delta(OD-a), \quad =0,$$

and the coefficient of $\sin\theta$ is

$$a\alpha p'+b\beta q'+c\gamma r',$$

where p', q', r' are the coordinates of the line DE; but the point α, β, γ lies on this line, and therefore the expression vanishes. We have therefore

$$OM(a\alpha+b\beta+c\gamma)=a\alpha p+b\beta q+c\gamma r,$$

where (α, β, γ), (p, q, r) stand for actual distances.

But the equation is homogeneous in α, β, γ, and we may therefore use it even when α, β, γ are simply proportional to actual distances; to ensure the same convenience as regards p, q, r, we make the equation homogeneous in p, q, r by means of the identical relation

$$a^2p^2+b^2q^2+c^2r^2$$
$$-2bq\,.\,cr\,.\cos A-2cr\,.\,ap\,.\cos B-2ap\,.\,bq\,.\cos C=4\Delta^2,$$

when it becomes

$$OM(a\alpha+b\beta+c\gamma)$$
$$\times(a^2p^2+b^2q^2+c^2r^2-2bq.cr.\cos A-2cr.ap.\cos B-2ap.bq.\cos C)^{\frac{1}{2}}$$
$$=2\Delta(a\alpha p+b\beta q+c\gamma r):$$

hence we have the result:—

The distance from the point α, β, γ to the line p, q, r is

$$\frac{2\Delta(a\alpha p+b\beta q+c\gamma r)}{(a\alpha+b\beta+c\gamma)(a^2p^2+b^2q^2+c^2r^2-2bq\,.\,cr\,.\cos A-2cr\,.\,ap\,.\cos B-2ap\,.\,bq\,.\cos C)^{\frac{1}{2}}}.$$

If we use more general point coordinates,

$$x:y:z=l\alpha:m\beta:n\gamma,$$

with the associated line system

$$\xi:\eta:\zeta=\frac{a}{l}p:\frac{b}{m}q:\frac{c}{n}r,$$

the expression for the distance from the point x, y, z to the line ξ, η, ζ becomes

$$\frac{2\Delta(x\xi+y\eta+z\zeta)}{\left(\frac{a}{l}x+\frac{b}{m}y+\frac{c}{n}z\right)(l^2\xi^2+m^2\eta^2+n^2\zeta^2-2mn\eta\zeta\cos A-2nl\zeta\xi\cos B-2lm\xi\eta\cos C)^{\frac{1}{2}}}.$$

The particular forms assumed when (1) *trilinears*, (2) *areals*, are used should be noticed.

Pole and Polar with regard to a Triangle.

22. It is usually convenient to determine the multipliers l, m, n so that a particular line of the diagram may have a particular equation, or so that a particular point may have particular coordinates.

Suppose *e.g.* we wish the centroid of the triangle to be 1, 1, 1. The actual distances being α, β, γ, we have $\alpha=\frac{1}{3}\cdot\frac{2\Delta}{a}$, etc.; we must therefore take $l\times\frac{2\Delta}{3a}=m\times\frac{2\Delta}{3b}=n\times\frac{2\Delta}{3c}$, in order that this point may have $x=y=z$. These equations give

$$l:m:n=a:b:c,$$

hence $$x:y:z=a\alpha:b\beta:c\gamma,$$

and we must use areals.

It should be noticed that the condition *a given point has specified coordinates* determines only the ratios $l:m:n$; it gives us a result of the form $x:y:z=A\alpha:B\beta:C\gamma$, and not $x=A\alpha$, etc. Practically, we hardly ever require actual coordinates; in fact, if we use the fundamental identical relations properly, we never require them. A point f, g, h may be taken as having coordinates kf, kg, kh, where k is any multiplier we please; this follows from the fact already noticed that all equations used can be made homogeneous.

23. Whatever system of coordinates we employ, the point 1, 1, 1 and the line 1, 1, 1,—or, more generally, the point f, g, h and the line $\frac{1}{f}$, $\frac{1}{g}$, $\frac{1}{h}$,—have a simple geometrical connection.

Let P be the point f, g, h; let AP, BP, CP meet BC, CA, AB in A', B', C'; and let $B'C'$ meet BC in A'', and similarly let B'', C'' be determined; then A'', B'', C'' are collinear, and the line they determine is the line $\frac{1}{f}$, $\frac{1}{g}$, $\frac{1}{h}$ (Fig. 9).

The lines AP, BP, CP have equations

$$\frac{y}{g}-\frac{z}{h}=0,\quad \frac{z}{h}-\frac{x}{f}=0,\quad \frac{x}{f}-\frac{y}{g}=0;$$

hence B' is the intersection of

$$y=0 \quad\text{and}\quad \frac{z}{h}-\frac{x}{f}=0.$$

Any line through B' has therefore an equation of the form

$$-\frac{x}{f}+my+\frac{z}{h}=0.$$

For this to pass through C' it must be of the form

$$-\frac{x}{f}+\frac{y}{g}+nz=0;$$

hence m must have the value $\frac{1}{g}$, and we find that $B'C'$ is

$$-\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0.$$

Now A'' is the intersection of this and $x=0$; hence any line through A'' has an equation of the form

$$Mx-\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0,$$

that is,
$$Ax+\frac{y}{g}+\frac{z}{h}=0.$$

For this to pass through B'', it must be of the form

$$\frac{x}{f}+By+\frac{z}{h}=0;$$

hence
$$A=\frac{1}{f},$$

and the line $A''B''$ is

$$\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0;$$

the form of this equation shows that the line passes through C''; the three points A'', B'', C'' therefore lie on the line

$$\frac{1}{f}, \frac{1}{g}, \frac{1}{h}.$$

The point and line, so related, are said to be pole and polar with regard to the triangle.

There is also a construction which starts from the line, and leads to the point.

Let p be the line f, g, h; let the join of bc (i.e. A) and ap be a', and similarly let b', c' be determined. Let the join of $b'c'$ and bc be a'', and similarly for b'', c''; then a'', b'', c'' are concurrent, and the point they determine is

$$\frac{1}{f}, \frac{1}{g}, \frac{1}{h} \text{ (Fig. 9).}$$

The points ap, bp, cp have equations

$$\frac{\eta}{g}-\frac{\zeta}{h}=0, \quad \frac{\zeta}{h}-\frac{\xi}{f}=0, \quad \frac{\xi}{f}-\frac{\eta}{g}=0;$$

hence b' is the join of

$$\eta=0 \quad \text{and} \quad \frac{\zeta}{h}-\frac{\xi}{f}=0.$$

Any point on b' has therefore an equation of the form

$$-\frac{\xi}{f}+m\eta+\frac{\zeta}{h}=0.$$

For this to lie on c' it must be of the form

$$-\frac{\xi}{f}+\frac{\eta}{g}+n\zeta=0;$$

hence m must have the value $\frac{1}{g}$, and we find that $b'c'$ is

$$-\frac{\xi}{f}+\frac{\eta}{g}+\frac{\zeta}{h}=0.$$

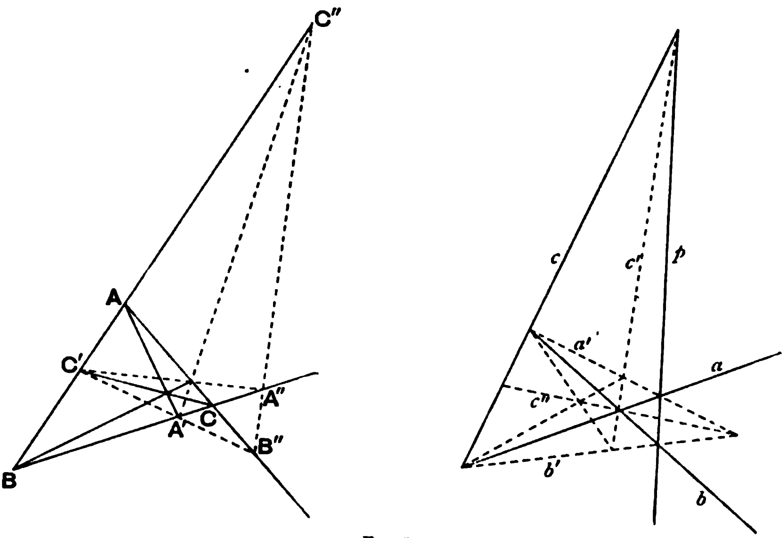

FIG. 9.

Now a'' is the join of this with $\xi=0$; hence any point on a'' has an equation of the form

$$M\xi-\frac{\xi}{f}+\frac{\eta}{g}+\frac{\zeta}{h}=0,$$

that is, $$A\xi+\frac{\eta}{g}+\frac{\zeta}{h}=0.$$

For this to lie on b'', it must be of the form

$$\frac{\xi}{f}+B\eta+\frac{\zeta}{h}=0;$$

hence $A=\frac{1}{f}$, and the point $a''b''$ is

$$\frac{\xi}{f}+\frac{\eta}{g}+\frac{\zeta}{h}=0:$$

the form of this equation shows that the point lies on c''; the three lines a'', b'', c'' therefore meet in the point $\frac{1}{f}, \frac{1}{g}, \frac{1}{h}$.

24. When the point 1, 1, 1 is known, or more generally when any particular point f, g, h is known, the system of coordinates is determined, so the positions of all other points are given. One way of determining the coordinates of any desired point or line in the diagram is the following:—

Let O be 1, 1, 1; with centre O and any convenient radius describe a circle to cut all the sides of the triangle. Take on each side one of the points so determined, D, E, F; then OD, OE, OF are equal; they therefore represent the coordinates of O. The coordinates of any other point P are represented by PL, PM, PN drawn parallel to OD, OE, OF.

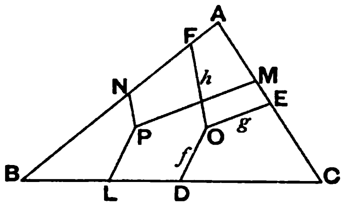

FIG. 10.

Similarly if O be the point f, g, h; determine points D, E, F so that OD, OE, OF represent f, g, h on any scale; the coordinates of any other point P are represented by PL, PM, PN drawn parallel to OD, OE, OF (Fig. 10).

So also a point with any assigned coordinates can be accurately inserted in the diagram; and any desired line can be drawn, by means of the points in which it meets the sides. A better method, however, follows from principles to be explained in Chapter III.

EXAMPLES.

1. Determine the distances to the sides of the triangle of reference from the following points:—(i.) the centroid; (ii.) the orthocentre; (iii.) the centre of the circumscribed circle; (iv.) the centre of the inscribed circle; (v.), (vi.), (vii.) the centres of the three escribed circles. Give the coordinates of these seven points (a) in trilinears, (b) in areals.

2. Determine the seven different systems of coordinates in which these points shall be 1, 1, 1; and determine in every case the associated system of line coordinates. Verify that the condition of collinearity is satisfied by the points (i.), (ii.), (iii.).

3. Determine the distances from A, B, C to
   (i.) the internal bisectors of the angles A, B, C;
   (ii.) the external bisectors of these angles;
   (iii.) the lines through A, B, C bisecting BC, CA, AB;
   (iv.) the lines through A, B, C perpendicular to BC, CA, AB.

Give the coordinates of these four sets of lines in the two line systems that are associated with (a) trilinears, (b) areals.

4. Having obtained the coordinates of these sets of lines, write down their equations. Verify that the condition of concurrence is satisfied by the lines in sets (i.), (iii.), (iv.).

5. Find the point coordinates of the intersection of the lines whose line coordinates are l_1, m_1, n_1, and l_2, m_2, n_2. State the result with reference to the equations of the lines and their intersection.

6. Show (a) by point coordinates,
   (b) by line coordinates,

that if the joins of vertices of two triangles be concurrent, then the intersections of sides are collinear.

Note. Triangles thus situated are said to be in *perspective;* the point and line are the centre and axis of perspective; or again, the relation of the triangles is spoken of as *homology;* the point and line are the centre and axis of homology.

7. Find the equations of lines through A, B, C making with AB, BC, CA (inside the triangle) angles ω. Find the value of ω if these be concurrent; and the coordinates of their common point.

Show that the same value of ω ensures the concurrence of lines through A, B, C making with AC, BA, CB angles ω.

Note. These two points are the Brocard points of the triangle; ω is the Brocard angle.

CHAPTER II.

INFINITY. TRANSFORMATION OF COORDINATES.

Parallel Lines.

25. Taking any two lines whose equations in point coordinates are

$$lx+my+nz=0, \quad l'x+m'y+n'z=0,$$

and solving for the coordinates of their point of intersection we find $\quad x:y:z=mn'-m'n:nl'-n'l:lm'-l'm.$

To obtain the actual values of the coordinates, we make use of the fundamental identical relation, which we suppose to be written in the form

$$a_0x+b_0y+c_0z=1.$$

This gives

$$x=\frac{mn'-m'n}{D}, \quad y=\frac{nl'-n'l}{D}, \quad z=\frac{lm'-l'm}{D},$$

where D stands for the determinant

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ a_0 & b_0 & c_0 \end{vmatrix}.$$

Thus we have ordinarily a set of finite values for x, y, z, giving for the intersection of the two lines a point at a finite distance from every side of the triangle of reference.

But if $D=0$, those fractions whose numerators do not also vanish become infinite. Now the vanishing of two of the numerators entails the vanishing of the third; for $mn'-m'n=0$ gives $\frac{m}{m'}=\frac{n}{n'}$, and $nl'-n'l=0$ gives $\frac{n}{n'}=\frac{l}{l'}$; hence $\frac{l}{l'}=\frac{m}{m'}$, that is, $lm'-l'm=0$. But these give simply $l:m:n=l':m':n'$, which would make the two lines identical. Hence not more than one of the numerators can vanish. Thus of the coordinates x, y, z, certainly two and possibly

all three are infinite, and the point x, y, z is at infinity. We see then that two lines may be so situated that instead of a finite intersection they have their intersection at infinity. This agrees with the properties of parallel lines that we have already employed; and by means of this we formulate the definition as follows:—

Lines that meet in a point at infinity are said to be parallel.

The condition of parallelism is therefore

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ a_0 & b_0 & c_0 \end{vmatrix} = 0,$$

where $a_0x+b_0y+c_0z=1$ is the fundamental identical relation.

The Special Line at Infinity.

26. The form of this condition recalls the condition that three lines be concurrent. It is in fact the condition for the concurrence of

$$lx+my+nz=0,$$
$$l'x+m'y+n'z=0,$$
$$a_0x+b_0y+c_0z=0;$$

that is, of the two given lines, and a line which is the same whatever pair (l, m, n), (l', m', n') be chosen. We naturally consider therefore what is the significance of this special line

$$a_0x+b_0y+c_0z=0.$$

One peculiarity in the equation is at once evident; it is at variance with the fundamental identical relation. Now this fundamental relation was obtained by a process certainly valid *for finite values of* α, β, γ; but not so obviously admissible if any of the three α, β, γ happen to be infinite. We shall find that this explains the apparent paradox; the line we are to consider, viz. $a_0x+b_0y+c_0z=0$, is a line lying entirely at infinity, and accounting entirely for infinity in point coordinates.

I. *Every point on this line is at infinity.* For to determine the actual coordinates of a point on the line, we have to determine x, y, z to satisfy the two equations

$$a_0x+b_0y+c_0z=0 \quad \text{..........................(1)},$$
$$a_0x+b_0y+c_0z=1 \quad \text{..........................(2)},$$

i.e. to satisfy the equation obtained by subtraction,

$$0\,.\,x+0\,.\,y+0\,.\,z=1 \quad \text{..........................(3)}.$$

Now (3) requires one at least of the quantities x, y, z to be infinite; and then, by equation (1), certainly one other must

be infinite; thus two of the coordinates of any point on the line are infinite, *i.e.* every point on the line is at infinity.

II. But further, *this line is the complete point representative of infinity;* that is, every point at infinity lies on it. For let X be a point at infinity, and let P, Q be points at a finite distance, chosen so that PQ does not pass through X. Let the equations of PX, QX be

$$lx+my+nz=0, \quad l'x+m'y+n'z=0.$$

These lines, meeting at X, are by definition parallel, and consequently

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ a_0 & b_0 & c_0 \end{vmatrix}=0;$$

that is, the three lines

$$lx+my+nz=0, \quad l'x+m'y+n'z=0, \quad a_0x+b_0y+c_0z=0$$

are concurrent, and consequently

$$a_0x+b_0y+c_0z=0$$

passes through X.

It appears as though this special line might be regarded as parallel to every line u, for it meets u at infinity, and therefore satisfies our definition of parallelism. This would also make it perpendicular to every line u, being parallel to a line perpendicular to u. But for reasons that will appear later, the idea of direction must not be associated with this line; any attempt at assigning direction to it leads to absolute indetermination.

27. Now the fundamental identical relation in point coordinates, being obtained from

$$a\alpha+b\beta+c\gamma=2\Delta,$$

may have for the constant on the right any finite quantity, though we generally arrange the equation so that the value of this constant may be unity. The relation is therefore essentially of the form

$$a_0x+b_0y+c_0z \neq 0;$$

hence we have the conclusion:—

In general the coordinates of a point in a plane are conditioned by a linear inequality $a_0x+b_0y+c_0z \neq 0$, *but there are exceptional points in the plane which make* $a_0x+b_0y+c_0z=0$; *these exceptional points are the totality of points on a certain straight line which lies entirely at infinity, and has for its equation* $a_0x+b_0y+c_0z=0$.

Note. It is well to notice that a similar statement may be made with reference to any linear function $u=lx+my+nz$. Taking any

point P at random, this does not lie on the line u; that is, in general the coordinates of a point in the plane make $u \neq 0$; but there are points for which $u=0$, viz., the totality of points lying on the line u. The bearing of this remark will appear in Chapter X.

28. Using trilinears, the line infinity is

$$ax+by+cz=0.$$

Now the expression for the distance from a point x, y, z to a line ξ, η, ζ has been shown to be

$$\frac{2\Delta(x\xi+y\eta+z\zeta)}{(ax+by+cz)(\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C)^{\frac{1}{2}}}.$$

When we compare distances from different points to a fixed line, ξ, η, ζ are constant quantities, and therefore

$$\frac{2\Delta}{(\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C)^{\frac{1}{2}}}$$

is simply a constant multiplier k. Thus the distance from a variable point x, y, z to a fixed line l, m, n is $k\dfrac{lx+my+nz}{ax+by+cz}$, where k depends on l, m, n. If then we take for l, m, n, λ multiplied by a, b, c, the numerator in this fraction becomes $\lambda(ax+by+cz)$, and consequently $ax+by+cz$ divides out. Thus the expression for the distance from any point x, y, z to the special line $ax+by+cz=0$ is $k\lambda$, where $k\lambda$ does not depend on x, y, z; consequently *all ordinary points in the plane must be regarded as being at an absolutely constant distance from the special line $ax+by+cz=0$.* It is obvious that this constant has not a *finite* value; and in fact k involves in its denominator the square root of

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C,$$

where ξ, η, ζ are respectively a, b, c. Now

$$a^2+b^2+c^2-2bc\cos A-2ca\cos B-2ab\cos C$$
$$=a^2+b^2+c^2-(b^2+c^2-a^2)-(c^2+a^2-b^2)-(a^2+b^2-c^2)=0,$$

and thus the absolute constant $k\lambda$ has an infinite value.

29. We have here shown that the line coordinates of the special line $ax+by+cz=0$ contradict the fundamental identical relation in line coordinates, for they make a certain expression vanish, while the identical relation asserts that this expression has a constant value different from zero. By what has been pointed out already regarding the applicability of the Principle of Duality, we know that we must not expect the above investigation to apply exactly to line coordinates; but we have here a hint that there

is an analogous paradox and presumably an explanation. To this we shall return when considering expressions of the second degree. (See Ch. VII.)

Relation of Cartesians and Homogeneous Coordinates.

30. With the help of the line infinity, Cartesians may be exhibited as a special case of homogeneous coordinates, and simple formulæ of transition can be found.

Taking the Cartesian axes, and adjoining to them this special line, we have three non-concurrent lines, which may

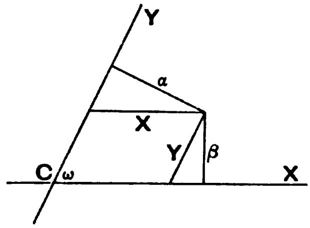

FIG. 11.

therefore be taken as lines of reference. Let the Cartesian coordinates be X, Y, and take the line $X=0$, *i.e.* the axis of Y, for the line $\alpha=0$; take $Y=0$ for $\beta=0$, and the line infinity for $\gamma=0$. Then we have, from Fig. 11,

$$X \sin \omega = \alpha, \quad Y \sin \omega = \beta:$$

and $$\gamma = \text{a constant (§ 28).}$$

Let the homogeneous coordinates be as usual

$$x, y, z = l\alpha, m\beta, n\gamma;$$

then $$x = l'X, \quad y = m'Y, \quad z = \text{constant}.$$

Now although γ is infinite, yet it has been shown to be an absolute constant; therefore by properly choosing n, ? $n\gamma$ (i.e. z) can be made to assume any finite value we please, *e.g.* unity; our formulæ of transition from homogeneous coordinates to Cartesians are therefore

$$x = \text{any multiple of } X,$$
$$y = \text{any (other) multiple of } Y,$$
$$z = \text{any convenient constant};$$

that is, if we choose, $x=X$, $y=Y$, $z=1$; and we pass from Cartesians to homogeneous coordinates by introducing in the various terms such powers of z as will make the equation homogeneous.

The line infinity is here $z=0$, *i.e.* $0.x+0.y+z=0$; which in Cartesians is

$$0.X+0.Y+C=0,$$

that is, $$C=0.$$

Thus the line infinity presents itself in Cartesians under the paradoxical form

$$\text{finite constant}=0.$$

31. The effect of this transformation on the implied line coordinates is noteworthy.

The points A, B are at infinity; let a line cut BC, CA, in L, M (Fig. 12).

Now MA and LB are distances from M, L to the line infinity, and we have seen that all such distances are to be considered equal. For the same reason CA and CB are equal.

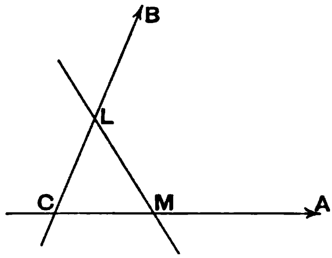

FIG. 12.

We have seen (§ 12) that $r:p=-CM:MA$,

$$r:q=-CL:LB;$$

hence $$-p.CM=r.MA,\quad -q.CL=r.LB;$$

that is, $$p:-\frac{1}{CM}=q:-\frac{1}{CL}=r:\text{constant}.$$

Hence the coordinates of the line are now represented by $-\frac{1}{CM}$, $-\frac{1}{CL}$, with any convenient constant, *e.g.* unity, for the third; and we see that the proper line coordinates to use in Cartesian geometry are the negative reciprocals of the intercepts made on the axes; a result which can also be obtained by the comparison of the two equations,

(1) $\frac{X}{a}+\frac{Y}{b}-1=0$, the Cartesian equation of a line in terms of the intercepts made on the axes;

(2) $x\xi+y\eta+z\zeta=0$, the relation adopted as controlling the choice of point and line coordinates.

32. We can however connect Cartesians and homogeneous coordinates without making use of the line infinity.

Suppose we have certain fixed lines a, b, c, etc., whose Cartesian equations in the standard form are

$$x \cos a + y \sin a - p_1 = 0, \quad x \cos \beta + y \sin \beta - p_2 = 0, \text{ etc.},$$

and let us use the letters a, β, etc., as abbreviations for $x \cos a + y \sin a - p_1$, etc., then $a = 0$ represents the line a, and the value of a at any point P gives the distance from P to the line a. The equation of any line can now be written in the form

$$la + m\beta + n\gamma = 0.$$

For

$$\left.\begin{aligned} a &\equiv a_1x + a_2y + a_3 \\ \beta &\equiv b_1x + b_2y + b_3 \\ \gamma &\equiv c_1x + c_2y + c_3 \end{aligned}\right\} \quad \text{.....................(1)},$$

and therefore

$$la + m\beta + n\gamma$$
$$= x(a_1l + b_1m + c_1n) + y(a_2l + b_2m + c_2n) + a_3l + b_3m + c_3n.$$

In order then to write $Ax + By + C$ in the form $la + m\beta + n\gamma$ we must have

$$a_1l + b_1m + c_1n = A,$$
$$a_2l + b_2m + c_2n = B,$$
$$a_3l + b_3m + c_3n = C,$$

equations which determine l, m, n uniquely unless the determinant $(a_1b_2c_3) = 0$. The exception refers to the case when the three selected lines a, b, c are concurrent, a case which we exclude by saying that the three lines must form a triangle.

So again even if the lines a, b, c be not in the standard form, there is a corresponding transformation. For using u, v, w as abbreviations for $a_1x + a_2y + a_3$, etc., u, v, w though not now giving the values of the perpendiculars, are certain multiples of them, and are therefore available as homogeneous coordinates.

Thus we can pass from Cartesians to a system of homogeneous coordinates with any desired triangle for triangle of reference; and from homogeneous coordinates to Cartesians. In particular, if the triangle of reference be that formed by the Cartesian axes and the line $aX + bY + c = 0$, then the formulæ of transformation are

$$x = \text{any multiple of } X,$$
$$y = \text{any multiple of } Y,$$
$$z = \text{any multiple of } aX + bY + c;$$

and taking advantage of the multipliers that are here at our disposal, we can if we choose write

$$x = X,$$
$$y = Y,$$
$$z = aX + bY + c.$$

There may be considerations influencing the choice of multipliers, so that this particular selection may possibly not be the best. This point is discussed in Chapter X.

The equations of transformation (1) being linear, the degree of the equation is unaltered when we pass from Cartesians to homogeneous coordinates.

Change of the Triangle of Reference.

33. Let it be required to change to a new triangle of reference; *e.g.* to the one determined by $u=0$, $v=0$, $w=0$, where

$$u = lx + my + nz,$$
$$v = l'x + m'y + n'z,$$
$$w = l''x + m''y + n''z.$$

Solving these equations, x, y, z are given in terms of u, v, w in the form

$$x \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} = \begin{vmatrix} u & m & n \\ v & m' & n' \\ w & m'' & n'' \end{vmatrix};$$

that is, $xD = Lu + L'v + L''w$, etc., where D is the *determinant of transformation*, or more often the *modulus of transformation;* the non-evanescence of D is secured by the condition that u, v, w form a triangle.

Since only the ratios of $x:y:z$ are required, the factor D may be dropped, and the resulting formulæ of transformation are

$$x = Lu + L'v + L''w,$$
$$y = Mu + M'v + M''w,$$
$$z = Nu + N'v + N''w.$$

These formulæ being linear, the degree of an equation in point coordinates is not affected by any change in the fundamental triangle.

Similarly the triangle of reference can be changed in line coordinates. Let the vertices of the new triangle be $\theta=0$, $\phi=0$, $\psi=0$, where

$$\theta = \lambda\xi + \mu\eta + \nu\zeta,$$
$$\phi = \lambda'\xi + \mu'\eta + \nu'\zeta,$$
$$\psi = \lambda''\xi + \mu''\eta + \nu''\zeta.$$

Solving these equations, ξ, η, ζ are found as linear functions of the new coordinates θ, ϕ, ψ.

34. But frequently the two sets of coordinates are in use together, so that formulæ are required for the change of the fundamental triangle in the double system; the two sets of coefficients (l, m, n, l', etc.), (λ, μ, ν, λ', etc.) are no longer independent.

Consider any line whose equation referred to the original triangle is $\xi x+\eta y+\zeta z=0$, so that ξ, η, ζ are the original coordinates of the line. The transformed equation is to be $\theta u+\phi v+\psi w=0$, since θ, ϕ, ψ are to be the new line coordinates. The expression of x, y, z in terms of u, v, w transforms $\xi x+\eta y+\zeta z=0$ into

$$\xi(Lu+L'v+L''w)+\eta(Mu+M'v+M''w)+\zeta(Nu+N'v+N''w)=0,$$

that is, into

$$(L\xi+M\eta+N\zeta)u+(L'\xi+M'\eta+N'\zeta)v+(L''\xi+M''\eta+N''\zeta)w=0.$$

Hence θ, ϕ, ψ must be proportional to

$$L\xi+M\eta+N\zeta, \text{ etc.}$$

Comparing the formulæ here obtained for the point and line transformations, it appears that only the one set of nine coefficients is involved; and that by means of these nine coefficients the old point coordinates are given in terms of the new, and the new line coordinates in terms of the old, the equations being

$$\begin{aligned} x&=Lu+L'v+L''w, & \theta&=L\xi+M\eta+N\zeta,\\ y&=Mu+M'v+M''w, & \phi&=L'\xi+M'\eta+N'\zeta,\\ z&=Nu+N'v+N''w; & \psi&=L''\xi+M''\eta+N''\zeta; \end{aligned}$$

which may also be written

$$u=lx+my+nz, \text{ etc.},$$
$$\xi=l\theta+l'\phi+l''\psi, \text{ etc.}$$

Two substitutions related as above are said to be *inverse;* one expresses a system of old variables in terms of new variables, the other expresses a related system of new variables in terms of old variables; the coefficients are the same, and if read in the one case by columns, and in the other case by rows, they are in the same order.

EXAMPLES.

1. Using areals, find the equation of the line bisecting two sides of the triangle of reference, and show that it is parallel to the third side.

2. Find (*a*) in trilinears, (*b*) in areals, the equation of the line through a vertex parallel to the opposite side.

3. Representing the line infinity in any system by $s=0$, prove that the four lines u, v, $u+ks$, $v+ls$ form a parallelogram, and find its diagonals.

Interpret this if $s=0$ be a line not at infinity.

4. Taking any four lines, no three of which are concurrent, as $x=0$, $y=0$, $z=0$, $u=0$, where $x+y+z+u\equiv 0$, determine what lines are represented by $x\pm y=0$, $x\pm z=0$, etc.

Construct the general diagram; and a special one for the case when x, y, z, are areal coordinates.

CHAPTER III.

FIGURES DETERMINED BY FOUR ELEMENTS.

Collinear Points and Concurrent Lines.

35. Among figures determined by assigned points or lines those in which the points or lines are collinear or concurrent are naturally considered first.

Points lying on a line are spoken of as a *range*, lines passing through a point are spoken of as a *pencil*. In considering a range quantitatively, we are not concerned with actual distances from point to point, but, for a reason whose full force appears in a later chapter (Ch. IX.), with functions of the ratios of these distances. Any such function depends on four points, taken in a determinate order. Let the four points be A, B, P, Q; consider the segment AB as divided, first by P, and consequently in the ratio $AP:PB$; secondly by Q, and consequently in the ratio $AQ:QB$. The ratio of these two ratios, that is,

$$\frac{AP}{PB} : \frac{AQ}{QB},$$

is called the Cross-ratio (Anharmonic Ratio, Doppelverhältniss) of the points; it is denoted by (AB, PQ), a symbol in which both the grouping and the order of the points are indicated; or by $\{APBQ\}$, in which, unless by a clearly specified convention, the grouping and order are not indicated.

36. To obtain the corresponding function for a pencil of four lines, let the rays a, b, p, q, meeting in O, be met by a transversal in points A, B, P, Q; then the cross-ratio of these points can be expressed in terms of the angles determined by the lines. For if h be the distance from O to the transversal, Fig. 13 shows that

$$\begin{aligned} h \,.\, AP &= 2 \times \text{area } OAP \\ &= OA \,.\, OP \,.\, \sin ap, \\ h \,.\, PB &= OP \,.\, OB \,.\, \sin pb, \end{aligned}$$

therefore $$\frac{AP}{PB}=\frac{OA}{OB}\cdot\frac{\sin ap}{\sin pb}.$$

Similarly $$\frac{AQ}{QB}=\frac{OA}{OB}\cdot\frac{\sin aq}{\sin qb},$$

and therefore $$\frac{AP}{PB}:\frac{AQ}{QB}=\frac{\sin ap}{\sin pb}:\frac{\sin aq}{\sin qb}.$$

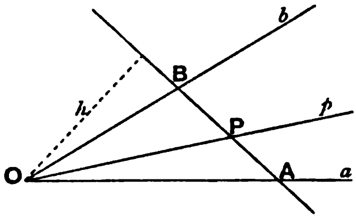

FIG. 13.

The cross-ratio determined by the pencil on the transversal is therefore the same for all positions of the transversal; this cross-ratio,

$$\frac{\sin ap}{\sin pb}:\frac{\sin aq}{\sin qb},$$

is called the cross-ratio of the pencil. It is denoted by $(ab,\ pq)$, or by $\{apbq\}$; or, when explicit reference to the vertex O is desired, by $\{O\,.\,APBQ\}$.

Hence, given a range, the cross-ratio of the pencil formed by taking as vertex any point in the plane is known, for it is equal to the cross-ratio of the range; and given a pencil, the cross-ratio of the range determined by any transversal is known, for it is equal to the cross-ratio of the pencil.

The Six Cross-ratios of Four Elements.

37. By taking the four given elements in different orders a number of different cross-ratios are obtained. There are twenty-four different orders, but the cross-ratios reduce to six different ones.

For, by definition, any one is the ratio of a product of segments $AP\,.\,QB$ to a complementary product $AQ\,.\,PB$. Now the four points determine six segments, and therefore three products of the admissible type. Writing these in the order

$$AB\,.\,PQ,\quad AP\,.\,QB,\quad AQ\,.\,BP,$$

it appears that any cross-ratio is the ratio of two of these,

taken with a negative sign. Hence calling these l, m, n, the six cross-ratios are

$$-\frac{m}{n},\quad -\frac{n}{l},\quad -\frac{l}{m};$$

$$-\frac{n}{m},\quad -\frac{l}{n},\quad -\frac{m}{l}.$$

But these six are not independent; they are in reciprocal pairs, and the product of the three in a row is -1. Moreover

$$\begin{aligned} l+m+n &= AB.PQ+AP.QB+(AP+PQ)(BQ+QP) \\ &= AB.PQ+AP.QB-(AP+PQ)(QB+PQ) \\ &= AB.PQ-PQ(AP+PQ+QB) \\ &= AB.PQ-PQ.AB=0. \end{aligned}$$

Hence $-\frac{m}{n}-\frac{l}{n}=1$, etc.; that is, if k be any cross-ratio, $1-k$ is also a cross-ratio.

The six members of a group of cross-ratios are, therefore,

$$k,\quad \frac{1}{1-k},\quad 1-\frac{1}{k},$$

$$\frac{1}{k},\quad 1-k,\quad -\frac{k}{1-k}.$$

Ex. Show that the permissible changes in the order of the elements determining any special cross-ratio are expressed by the formula:—Any two elements may be interchanged, provided that the other two be interchanged also.

38. An important special case is that of harmonic division, where AB is divided internally and externally in the same ratio; hence $AP:PB$ and $AQ:QB$ are equal in value but opposite in sign. This gives $(AB, PQ)=-1$. The scheme just given for the cross-ratios now becomes

$$-1,\quad \tfrac{1}{2},\quad 2,$$

$$-1,\quad 2,\quad \tfrac{1}{2}.$$

Hence $(PQ, AB)=-1$, which shows that if AB be harmonically divided by PQ, then PQ is harmonically divided by AB. The harmonic relation therefore involves the two pairs of points symmetrically.

39. When one of the four points (*e.g.* Q) is at infinity, any cross-ratio reduces to a simple ratio. For $\frac{AP}{PB}:\frac{AQ}{QB}$ becomes $\frac{AP}{PB}:-1$, since AQ, QB are equal in magnitude (being infinite)

but opposite in direction. Thus $(AB, P\infty)=AP:BP$. If now the pairs AB, PQ be harmonic, this gives us

$$(AB, P\infty)=-1;$$

that is, $$AP:PB=+1,$$

showing that AB is bisected at P. The relation thus exhibited between bisection and harmonic division has important consequences later, for it brings a certain class of metric properties within the legitimate application of homogeneous coordinates.

40. It has been shown that any line through the intersection of $u=0$, $v=0$ has an equation of the form $u-kv=0$; and that the value of u at any point P is some multiple of the perpendicular from P to the line $u=0$, that is, $u=l\alpha$, and similarly $v=m\beta$. Writing p for $u-kv$, Fig. 14 shows that

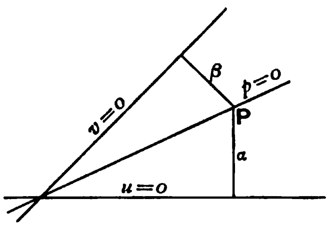

Fig. 14.

$$\frac{\sin up}{\sin pv}=\frac{\alpha}{\beta}$$
$$=\frac{m}{l}\cdot\frac{u}{v}$$
$$=\frac{m}{l}k,$$

since at P, $u-kv=0$.

Similarly taking another line $q=u-k'v=0$,

$$\frac{\sin uq}{\sin qv}=\frac{m}{l}k'.$$

Hence $$\frac{\sin up}{\sin pv}:\frac{\sin uq}{\sin qv}=k:k',$$

that is, the cross-ratio of the pencil formed by the two pairs $u=0$, $v=0$; $u-kv=0$, $u-k'v=0$; is $k:k'$. Now the equations of any four concurrent lines can be thrown into this form; hence the cross-ratio is known. Since $l:m$ does not appear in the result, it is immaterial what values the multipliers may have in the expressions $u=l\alpha$, $v=m\beta$.

The pairs of lines will be harmonic if $k:k'=-1$, that is, if their equations can be reduced to $u=0$, $v=0$; $u\pm kv=0$.

Similarly if $\theta=0$, $\phi=0$ be any two points, H, K; it has been shown that $\theta-k\phi=0$ is a point on the line $\theta\phi$; call the point P. Taking any line through P, draw the perpendiculars from H, K; these are proportional to θ, ϕ, that is,

$$\theta=l\,.\,HM,\quad \phi=m\,.\,KN.\quad \text{(Fig. 15.)}$$

Hence the equation of P, viz.,

$$\varpi=\theta-k\phi=0,$$

gives $$l\,.\,HM-km\,.\,KN=0.$$

But $$HM:KN=HP:KP,$$

therefore $$HP:KP=km:l,$$

whence $$\frac{HP}{PK}=-\frac{m}{l}k.$$

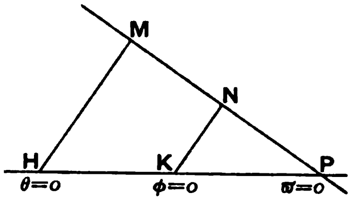

FIG. 15.

Similarly taking another point R, given by $\rho=\theta-k'\phi=0$,

$$\frac{HR}{RK}=-\frac{m}{l}k'.$$

Hence $$\frac{HP}{PK}:\frac{HR}{RK}=k:k';$$

and the cross-ratio of the range formed by the two pairs $\theta=0, \phi=0$; $\theta-k\phi=0, \theta-k'\phi=0$; is $k:k'$. Now the equations of any four collinear points can be thrown into this form, and thus their cross-ratio is known.

Hence if u, v be linear functions, whether in point or line coordinates, the pairs $u=0, v=0$; $u+kv=0$, $u+k'v=0$; give a configuration whose cross-ratio is $k:k'$. Now all four expressions are here of the same type, $u+\lambda v$, λ having the values $0, \infty, k, k'$. The result may therefore be stated in the symbolical form

$$(0\infty, kk')=k:k';$$

or, by means of the equalities among the twenty-four cross-ratios (see § 37),

$$(kk', 0\infty)=k:k' \text{ (compare § 39).}$$

41. The pairs of elements discussed in the last section were supposed given by their equations. Passing to the expression by means of coordinates, let

$$u=f_1x+g_1y+h_1z, \quad (\text{or } f_1\xi+g_1\eta+h_1\zeta),$$
$$v=f_2x+g_2y+h_2z, \quad (\text{or } f_2\xi+g_2\eta+h_2\zeta),$$

then u, v are respectively f_1, g_1, h_1; f_2, g_2, h_2;

$$u+kv \text{ is } f_1+kf_2,\quad g_1+kg_2,\quad h_1+kh_2;$$
$$u+k'v \text{ is } f_1+k'f_2,\quad g_1+k'g_2,\quad h_1+k'h_2;$$

and the result may be stated:—

The pairs of elements

$$(f_1, g_1, h_1),\quad (f_2, g_2, h_2);$$
$$(f_1+kf_2, g_1+kg_2, h_1+kh_2),\quad (f_1+k'f_2, g_1+k'g_2, h_1+k'h_2);$$

determine a configuration of cross-ratio $k:k'$.

And here, just as in § 40, the result may be stated symbolically:—

The cross-ratio of the configuration $(kk', 0\infty)$ *is* $k:k'$.

One particular aspect of the one-dimensional geometry here considered deserves special mention. Instead of considering points in a plane, limited to a line of that plane, and assigned by three homogeneous coordinates, we may confine ourselves to the one line on which the points lie. Let A, B be two fixed points on this line: the position of a variable point P depends on the ratio of AP to PB, *i.e.* if AP be called x, and PB y, on $x:y$; and we have a system of two homogeneous coordinates for the geometry of points on a line. Now let the segment P_1P_2 be divided in the ratio $\lambda:1$ by the point P;

then $$\frac{x-x_1}{x_2-x}=\lambda, \quad \text{and} \quad \frac{y-y_1}{y_2-y}=\lambda,$$

therefore $$x:y=x_1+\lambda x_2 : y_1+\lambda y_2.$$

Thus the conclusions of §§ 40, 41 apply to this case.

Ex. Show that the cross-ratio of the configuration (kk', ll') is $\frac{k-l}{l-k'}:\frac{k-l'}{l'-k'}$, whether we are dealing with equations or coordinates.

The Complete Quadrilateral and Quadrangle.

42. The correspondence between point and line configurations is gradually exhibiting itself as depending on a double interpretation of sets of quantities, and hence of equations. Coordinates may be interpreted as referring to points or to lines; equations of the first degree then represent lines or points; the cross-ratio of four elements whose equations are of a particular form can be determined without any knowledge as to whether the elements are lines or points. Thus by this dual interpretation a piece of algebraic work can be made to prove two theorems at once. This may be illustrated by certain theorems that will now be given relating to configurations determined by four non-concurrent lines or non-collinear points.

43. Four lines determine a quadrilateral, sometimes called a four-side; the intersection of any two sides is a vertex, there are therefore six vertices; as each vertex accounts for two sides, the six vertices fall into three pairs, the joins of these pairs are the three diagonals, forming the diagonal triangle.

Let the sides be (1), (2), (3), (4); call the vertices (12), (13), (14), D, E, F, and the complementary vertices (34), (24), (23), G, H, K; then the three diagonals a, b, c are DG, EH, FK. The entire configuration is spoken of as a complete quadrilateral; it is interesting on account of its *harmonic properties*. These are expressed by the theorem:—

Any diagonal is harmonically divided by the other two.

Note. The diagonal here referred to as having a determinate length is the segment determined by two vertices.

To prove this, take abc as triangle of reference, and choose coordinates so that the equation of the side (1) may be $x+y+z=0$. Then (2), (3), (4) have equations of the form

$$lx+y+z=0, \quad x+my+z=0, \quad x+y+nz=0;$$

but as (3), (4) are to meet on $x=0$, we must have $mn=1$, and similarly $nl=1$, $lm=1$; consequently

$$lmn=\pm 1.$$

If we choose the upper sign, we find $l=1$, $m=1$, $n=1$, values which make (2), (3), (4) the same as (1); the proper solution is therefore $lmn=-1$, whence

$$l=-1, \quad m=-1, \quad n=-1.$$

The sides are therefore

$$\begin{array}{ll} (1) & x+y+z=0, \\ (2) & -x+y+z=0, \\ (3) & x-y+z=0, \\ (4) & x+y-z=0. \end{array}$$

Note. We have here proved that by a proper choice of point coordinates the equations of four lines no three of which are concurrent can be thrown into the form $\pm x \pm y \pm z=0$; and therefore also that by a proper choice of line coordinates, the coordinates of the four lines can be made to be ± 1, ± 1, ± 1.

The equation of AD is at once found (Fig. 16); this is the line joining A to the intersection of (1), (2); it is therefore obtained by the elimination of x from the equations

$$x+y+z=0, \quad -x+y+z=0,$$

that is, AD is $y+z=0$; and similarly AG is $y-z=0$. These two lines are harmonic with regard to $y=0$, $z=0$; that is, DG is divided harmonically by BC.

44. A complete quadrangle (four-point) is determined by four points (1), (2), (3), (4); these joined in pairs (12), (13), (14); (34), (24), (23) give six sides d, e, f; g, h, k;

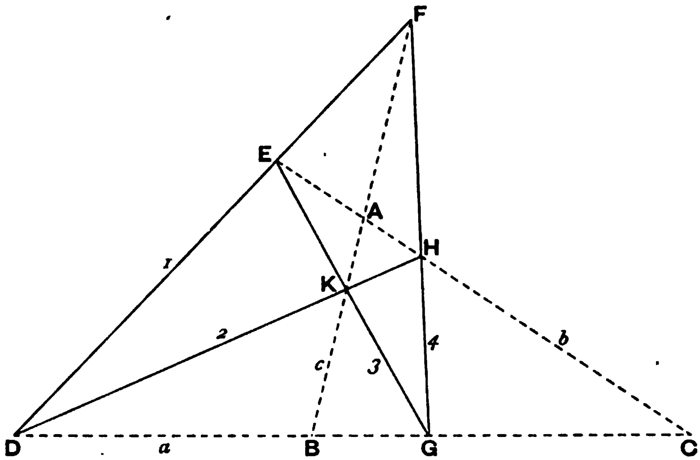

FIG. 16.

these in pairs determine three diagonal points A, B, C, forming the diagonal triangle. In this diagram the line-pair dg is harmonic with regard to bc (Fig. 17). This is

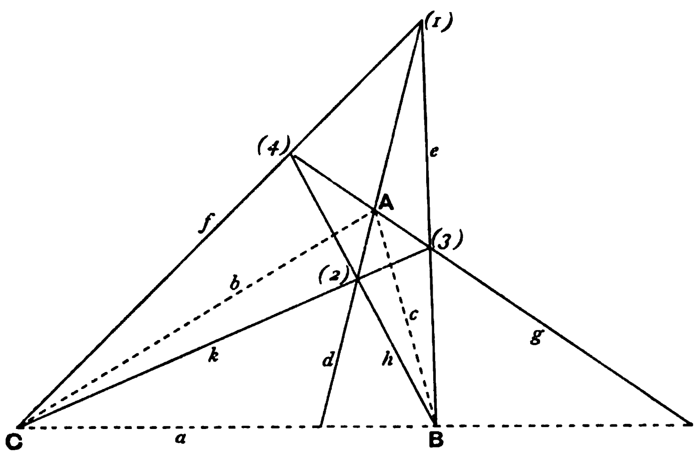

FIG. 17.

proved by exactly the work already used, the x, y, z now being interpreted as line coordinates; and incidentally it is

proved that the equations of four points can be made to be $\pm\xi\pm\eta\pm\zeta=0$, and that the coordinates of four points can be made to be ± 1, ± 1, ± 1.

To make this dual interpretation clear, the algebraic work is now written in an abbreviated form, with the two interpretations at the two sides. Symbols I., II., III. are used in the two cases for the vertices and the sides of the triangle of reference.

Comparison of Figs. 16 and 17 shows that the *fact* expressed by the two theorems is the same.

	(12) (34); (13) (24); (14) (23) determine the triangle of reference.	
DEF DKH GKE GHF	Let equation of (1) be $x+y+z=0$, then (2) is $lx+y+z=0$, (3) is $x+my+z=0$, (4) is $x+y+nz=0$.	def dkh gke ghf
(3) and (4) are to meet on the line a.	(3) and (4) are to give $x=0$; $\therefore mn=1$. Similarly $nl=1$, $lm=1$; $\therefore (lmn)^2=1$.	The join of (3), (4) is to pass through A.
	The solution $lmn=+1$ is inadmissible, $\therefore lmn=-1$, and therefore $l=m=n=-1$.	
	Hence (1) is $x+y+z=0$, (2) is $-x+y+z=0$, (3) is $x-y+z=0$, (4) is $x+y-z=0$.	
AD AG	Hence I. (12) is $y+z=0$, I. (34) is $y-z=0$,	ad ag
$\therefore D, G$ are harmonic with regard to B, C.	$\therefore$ (12), (34) are harmonic with regard to II., III.	$\therefore d, g$ are harmonic with regard to b, c.

On account of the harmonic properties here proved, the diagonal triangle is called the *harmonic triangle* of the quadrilateral or quadrangle.

45. The theorem just proved affords a construction by the ruler only for the fourth point Q in a harmonic range, when A, B, and P are given:—

Draw through A any two lines, cutting any line through B in (1), (2). Join P to either of these points, (1), and let this join meet the second line through A in (3); let B(3) meet the first line through A in (4); then (24) passes through Q.

Other constructions may be derived; the essential thing in all is to construct a quadrilateral with two vertices at A, B, and one diagonal through P; the other diagonal determines Q.

46. By means of the conception of cross-ratio we can insert accurately any line we please. To determine $lx+my+nz=0$, it suffices to find the points in which this line meets two sides, *e.g.* $x=0$, $y=0$. It meets $x=0$ where $my+nz=0$; thus we simply require to know how to draw any line $z=\lambda y$. Our choice of coordinates has virtually determined one line $z=\lambda_0 y$; for if the point O be 1, 1, 1, then AO is $z=y$. Let therefore AD (Fig. 18) be a known line $z=\lambda_0 y$; it is required to insert $z=\lambda y$.

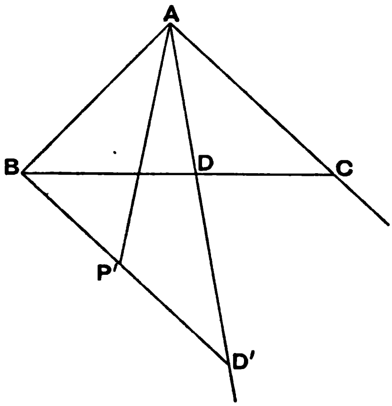

FIG. 18.

Draw from B a parallel to AC, meeting the known line in D'. Then consider a pencil with vertex A, and three known rays through B, C, D; the transversal cuts these in B, ∞, D'. Mark off on BD' a length BP', determined by $BP' : BD' = \lambda : \lambda_0$. Then $(B\infty, P'D') = BP' : BD' = \lambda : \lambda_0$.

If then AP' meet BC in P,

$$(BC, PD) = (B\infty, P'D') = \lambda : \lambda_0,$$

whence it follows that AP' is $z=\lambda y$.

Any point can be inserted by means of the lines joining it to any two vertices of the triangle.

Pairs of Points, Harmonically related.

47. The harmonic relation is essentially a relation between two pairs of elements,—for definiteness, between two pairs of points. It may also be regarded as a relation between two segments, each determined by one pair of points. Points on a line, represented by their distances from a fixed origin on the line, can be represented by an equation, non-homogeneous in one variable x; or again by an equation, homogeneous in two variables x, y. A pair of points is thus given by a quadratic equation.

Note. In saying that a group of n points is given by an equation of degree n, we imply that no distinction is made in the treatment of the points. If they are to be treated separately, they must be given by separate linear equations.

Consider two pairs on a line, given by the two quadratics

$$ax^2+2bx+c=0, \quad a'x^2+2b'x+c'=0.$$

These will not be harmonic unless a certain condition be satisfied; this, expressed geometrically, is

$$\frac{AP}{PB}:\frac{AQ}{QB}=-1.$$

Let the points be at distances from O equal to x_1, x_2; x_1', x_2'. This condition becomes

$$\frac{x_1'-x_1}{x_2-x_1'}\times\frac{x_2-x_2'}{x_2'-x_1}=-1,$$

that is, $$2x_1x_2-(x_1+x_2)(x_1'+x_2')+2x_1'x_2'=0,$$

whence $$ac'+a'c-2bb'=0.$$

If then the first pair AB be given, and we assume any point P as one of the second pair, the remaining point Q is determined linearly; we have a series of pairs P, Q; P', Q'; etc.; all harmonic with regard to A, B.

Ex. 1. Let $x^2-36=0$ give the first pair; and let $x-2=0$ give P, $x-h=0$ give Q. The harmonic condition just found shows that $h=18$.

Ex. 2. Let the bisection of PQ be $x=4$, the points A, B being given as in Ex. 1.

Then the quadratic for PQ is $a'x^2+2b'x+c'=0$, with the conditions

$$\frac{b'}{a'}=-4, \quad c'-36a'=0,$$

hence the quadratic is $x^2-8x+36=0$, an equation with imaginary roots. Thus a pair of imaginary values is found to be harmonic with regard to a given pair of real values.

Imaginary Elements.

48. This confronts us with the question of imaginary elements. Are they to be admitted? If so, on what terms?

Imaginaries present themselves naturally in the solution of algebraic equations, and are then recognized for the sake of continuity. If now we refuse to admit them into algebraic geometry, we shall have to examine the work at every step, to see whether it has a legitimate application in geometry; our symbolical language will no longer have an exact relation to the subject matter.

But there are even more cogent reasons why we should recognize imaginaries in algebraic geometry. One of the fundamental principles of the subject is that it takes two equations to represent a point; *one* equation can represent

only a locus (*i.e.* in point coordinates). But if we confine ourselves to real points, we have to say that such an equation as $x^2+y^2=0$ represents the origin only, that is, one equation represents a point. The alternative is to admit imaginary elements, and say that this equation represents the two imaginary lines $x+iy=0$, $x-iy=0$, these having for their intersection the real point, the origin (Salmon's *Conic Sections*, §§ 73 and 82).

We choose this alternative, and recognize imaginary elements, that is, elements whose coordinates have imaginary values, or whose equations have imaginary coefficients.

Note. The introduction of imaginary elements into Pure Geometry depends on different considerations, and requires independent justification.

49. Using trilinears for definiteness, let $f+if'$, $g+ig'$, $h+ih'$ be values of α, β, γ that satisfy the fundamental identical relation

$$a\alpha+b\beta+c\gamma=2\Delta.$$

The real and imaginary parts give

$$af+bg+ch=2\Delta \quad\ldots\ldots\ldots\ldots(1),$$

$$af'+bg'+ch'=0 \quad\ldots\ldots\ldots\ldots(2).$$

Of these, (2) shows that if two of the three quantities f', g', h' vanish, the third does also; that is, when two perpendiculars are real, all three are real. Hence one may be real, and two imaginary; or all three may be imaginary. But the coordinates can always be written as if one were real, for we are at liberty to multiply the three by any quantity we please.

Ex. $\qquad a=3,\ \ b=4,\ \ c=5;$ whence $2\Delta=12$.

The equations to be satisfied are

$$3f+4g+5h=12,$$

$$3f'+4g'+5h'=0.$$

One set of values satisfying these is

$$4-4i,\quad 5+8i,\quad -4-4i;$$

multiplying by $1+i$, these become

$$8,\quad -3+13i,\quad -8i;$$

which may therefore be taken for the coordinates of the imaginary point.

50. I. *Through every imaginary point there passes one real line.*

Note. It is plain that there cannot be two, for the intersection of two real lines is a real point.

Let the point be $f+if'$, $g+ig'$, $h+ih'$; the line

$$l\alpha+m\beta+n\gamma=0$$

will pass through this if

$$lf+mg+nh=0,$$
$$lf'+mg'+nh'=0;$$

that is, if $\quad l:m:n=gh'-g'h:hf'-h'f:fg'-f'g.$
Thus the line is determined uniquely.

Ex. On every imaginary line there is one real point.

II. *A real line contains an indefinite number of imaginary points, arranged in conjugate pairs.* For $f+if'$, $g+ig'$, $h+ih'$ lies on the real line $l\alpha+m\beta+n\gamma=0$ if

$$lf+mg+nh+i(lf'+mg'+nh')=0.$$

Hence f, g, h, f', g', h' must be determined to satisfy

$$\begin{aligned} lf+mg+nh&=0, \\ af+bg+ch&=2\Delta: \\ lf'+mg'+nh'&=0, \\ af'+bg'+ch'&=0. \end{aligned}$$

But these conditions being satisfied, not only will $f+if'$, $g+ig'$, $h+ih'$ lie on the line, but also $f-if'$, $g-ig'$, $h-ih'$; thus the points are in conjugate pairs. That the number is indefinite appears from the fact that we have six quantities wherewith to satisfy four equations.

III. *Imaginaries that present themselves through real algebraic equations are in conjugate pairs;* for $f+if'$ being a root of a real equation, $f-if'$ is also a root. A pair of imaginaries always means a conjugate pair, not simply any two; the line joining a pair of imaginary points (or the intersection of a pair of imaginary lines) is necessarily real, its equation being

$$\alpha(gh'-g'h)+\beta(hf'-h'f)+\gamma(fg'-f'g)=0.$$

IV. *Conjugate imaginary lines pass through conjugate imaginary points;* for if $f+if'$, $g+ig'$, $h+ih'$ satisfy $u+iv=0$, then $f-if'$, $g-ig'$, $h-ih'$ must satisfy $u-iv=0$.

51. Just as four real points determine a quadrangle, four imaginary points may be taken as determining elements; in fact, the investigations in §§ 43, 44 apply whether the elements be real or imaginary. But it is of some importance to know how the real and imaginary parts in the resulting configuration are arranged.

Let the quadrangle be determined by two pairs of imagin-

ary points, AA', BB'; the lines AA', BB' are real, and no other real line can go through these imaginary points. Consequently the four cross-joins AB, etc., are imaginary; but they are in conjugate pairs, viz., AB, $A'B'$ form one pair, and give therefore a real intersection; similarly for AB', $A'B$. Thus the complete quadrangle determined by two pairs of imaginary points has its sides, one pair real, two pairs imaginary; and the diagonal triangle is real. In Fig. 19 real lines are represented by solid lines, conjugate imaginaries by the same kind of broken line; the diagonal points are P, Q, R.

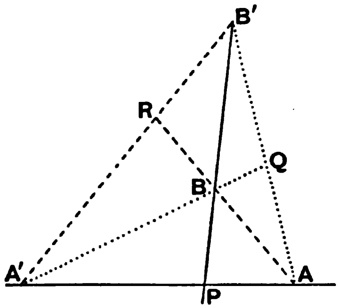

FIG. 19.

Similarly for the complete quadrilateral determined by two pairs of imaginary lines. Fig. 19 represents this also; the given pairs are QB, QB' and RB, RB'; the real vertices are Q, R; the pairs of imaginary vertices are AA', BB'. The three diagonal lines QR, AA', BB' are real.

Ex. **Discuss the configuration when the determining elements are one pair real, and one pair imaginary.**

52. It was proved in §§ 43, 44 that the coordinates of four elements can be made to be 1, ± 1, ± 1 by a suitable choice of coordinates, the diagonal triangle being taken for triangle of reference. If the elements be imaginary, this involves the use of imaginary multipliers l, m, n in the relations $x = la$, etc. This, while causing no difficulty in general, is inadmissible if we wish to discriminate between real and imaginary in the final results. We have seen that if the four elements be two imaginary pairs, the diagonal triangle is real, and can therefore be taken as triangle of reference. Let the given elements be points, for definiteness, viz., the pairs AA', BB' in Fig. 19; then PQR is to be taken as triangle of reference. The lines AA', BB' are real, and they have been shown to be harmonic with regard to PR, PQ; if

then coordinates y, z be chosen so that AA' is $y+z=0$, BB' is $y-z=0$. Hence the coordinates of A being f, g, h, we have $g+h=0$; thus A is $f, g, -g$, where f, g are imaginary. Hence dividing throughout by g, A is $\lambda+i\mu, 1, -1$; and A', being conjugate to A, is $\lambda-i\mu, 1, -1$. Now QA, QA' are conjugate imaginary lines, harmonic with regard to QR, QP; their equations must therefore be of the forms $x \pm kiz=0$; comparing these with the actual forms

$$x+(\lambda+i\mu)z=0,$$
$$x+(\lambda-i\mu)z=0,$$

it appears that $\lambda=0$. Hence the points A, A' are $\pm i\mu, 1, -1$; and the numerical multiplier involved in the x being still undetermined, can be used so as to give to μ the value unity; the points A, A' are now $\pm i, 1, -1$. The point B is the intersection of AR (*i.e.* $x-iy=0$) and BB' (*i.e.* $y-z=0$); hence B is $+i, 1, 1$; and B' is $-i, 1, 1$. Thus without obliterating the distinction between real and imaginary, the coordinates of two pairs of imaginary elements can be made to be $\pm i, 1, \pm 1$; or, if preferred, $i, \pm 1, \pm 1$.

Examples.

1. If $(AB, PQ)=-1$, then $\dfrac{1}{AP}+\dfrac{1}{AQ}=\dfrac{2}{AB}$.

2. Any two lines are harmonic with regard to the bisectors of their angles.

3. If two equal ranges have one point in common, the joins of the other pairs of points are concurrent.

4. If two equal pencils have one ray in common, the intersections of the other pairs of rays are collinear.

5. Let the sides BC, CA, AB of a triangle be divided in the ratios $h:1$, $k:1$, $l:1$; then the points of division are collinear if $hkl=-1$; and the lines joining the points of division to the vertices are concurrent if $hkl=+1$.

6. The bisections of the diagonals of a complete quadrilateral are collinear.

7. The six lines joining the vertices of the diagonal triangle to the vertices of the complete quadrilateral (determined by (1), (2), (3), (4)) are concurrent in threes. Calling the four points of concurrence $\alpha, \beta, \gamma, \delta$, show that $\alpha, \beta, \gamma, \delta$ are the poles of (1), (2), (3), (4) with regard to the diagonal triangle.

8. Show by constructing the diagram that the diagonal triangle of the complete quadrilateral (1)(2)(3)(4) is the diagonal triangle of the complete quadrangle $\alpha\beta\gamma\delta$.

9. State, for the complete quadrangle, the theorem corresponding to that of Ex. 7; and show by means of the diagram in Ex. 8 that it has already been virtually proved.

10. Show that the four points (lines)

$$g-h,\ h-f,\ f-g;\quad g-h,\ -(h+f),\ f+g;$$
$$g+h,\ h-f,\ -(f+g);\quad -(g+h),\ h+f,\ f-g;$$

are collinear (concurrent): and that the cross-ratios of the configuration are the six expressions

$$\frac{f^2-g^2}{f^2-h^2},\ \text{etc.}$$

11. Find the equations of the lines joining f, g, h to the four points 1, ± 1, ± 1; and determine the cross-ratios of the pencil.

12. If the four points be l, $\pm m$, $\pm n$, find the lines joining them to f, g, h; and the cross-ratios of the pencil.

13. Show that the pencil determined at f, g, h by 1, ± 1, ± 1, is equal to the pencil determined at p, q, r by a, $\pm b$, $\pm c$, and to the range determined on the line p, q, r by the lines a, $\pm b$, $\pm c$, if the following relations hold;

$$a+b+c=0,$$
$$af^2+bg^2+ch^2=0,$$
$$bcp^2+caq^2+abr^2=0;$$

and that the six cross-ratios are

$$-\frac{b}{c},\ -\frac{c}{a},\ -\frac{a}{b};\quad -\frac{c}{b},\ -\frac{a}{c},\ -\frac{b}{a}.$$

CHAPTER IV.

THE PRINCIPLE OF DUALITY.

Correspondence hitherto noted.

53. In the preceding sections attention has continually been drawn to the correspondence between the geometrical theories in which the element is (i.) the point; (ii.) the straight line. This correspondence is a manifestation of the Principle of Duality; this appears more clearly if we speak of the primary element and the secondary element. In the one system these are to be interpreted as point and line; in the other as line and point.

The point......................The primary element.......................The line
has three coordinates x, y, z.

Two points.................. Two primary elements ....Two lines
determine
a line........................... a secondary element.......................a point.

United with this
line............................... secondary element ....point
we have an indefinite number of
points,.......................... primary elements,.......................lines,
but the coordinates of all these
satisfy one equation of the first degree,
called the
equation of the line....$\left\{\begin{matrix}\text{equation of the secondary}\\ \text{element.}\end{matrix}\right\}$ equation of the point.

The line.....................The secondary element......................The point
determined
by x_1, y_1, z_1; x_2, y_2, z_2 has for its equation

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

The condition that
three points................ three primary elements............three lines
lie on................................be united with...................pass through
a line..........................one secondary element.............a point
is $(x_1 y_2 z_3) = 0$.

Any point	Any primary element	Any line
lying on the line	belonging to the secondary element	passing through the point
	determined by x_1, y_1, z_1; x_2, y_2, z_2;	
	has coordinates $x_1+\lambda x_2$, $y_1+\lambda y_2$, $z_1+\lambda z_2$;	
	and if we take another	
such point	such primary element	such line
	the cross-ratio of the	
range	configuration	pencil
	is $\lambda : \lambda'$.	

A point	A primary element	A line
lying on	united with	passing through
a fixed line	a fixed secondary element	a fixed point
	has one degree of freedom;	
it can move	it can move	it can rotate
along the line	in the secondary element.	about the point.

Curves in the two Theories.

54. We now consider how curves present themselves in these two theories.

The general idea of a curve is that it is a succession of points arranged according to some law, so that after every point there is one next point; ordinarily the point after A, B does not lie on the straight line AB, that is, ordinarily not more than two consecutive points lie on a straight line. This law is algebraically expressed by an equation $F(x, y, z)=0$; and if F be of degree n, any straight line in the plane meets the curve in n points, and the curve is said to be of order n. The line joining any point A to the consecutive point B is a tangent to the curve; consecutive tangents AB, BC intersect in a point B on the curve.

Stating this in general terms, there is a succession of primary elements arranged according to some law, so that after every primary element there is one next element; ordinarily not more than two consecutive primary elements belong to one secondary element. This law is algebraically expressed by an equation in the coordinates of the primary element; this equation being of degree n, a secondary element has in common with the system n primary elements. The derived secondary element determined by two consecutive primary elements is closely related to the system; two consecutive derived secondary elements determine a primary element of the system.

Translating this into terms of lines and points, there is a succession of lines arranged according to some law, so that after every line there is one next line; ordinarily not more than two consecutive lines pass through a point. This law is algebraically expressed by an equation

$F(\xi, \eta, \zeta)=0$; and if F be of degree n, through any point in the plane there pass n lines of the system. The point determined by two consecutive lines of the system is closely related to the system; two consecutive points determine a line of the system.

It appears therefore that the lines of the system are regarded as tangents to some curve, their envelope, the points of this curve being determined as the intersections of consecutive tangents. The number of tangents from a point being n, the envelope is of class n. Thus the locus of order n, and the envelope of class n, are corresponding conceptions. For instance, if n be unity, the locus of order n is a straight line, and the envelope of class n is a point; the conception of the straight line in the point theory (*i.e.* of the straight line quà locus of points) corresponds to the conception of the point in the line theory (*i.e.* of the point quà envelope of lines).

55. Since the system of primary elements affords a derived system of secondary elements, from which again the primary elements may be derived, it follows that a curve may be considered under either aspect. It has points (*ineunts*, Cayley) from which the tangents may be derived, viz., by joining every point to the next; it has lines (*tangents*) from which the points may be derived, viz., by marking the intersection of every line with the next. The point system or the line system may be regarded as the primary system; the other is then the derived secondary system; thus the curve may be regarded as a locus or as an envelope. Every curve has therefore two different equations, one in point coordinates, $F(x, y, z)=0$, and one in line coordinates, $\Phi(\xi, \eta, \zeta)=0$; if these be respectively of degrees m and n, the curve is of order m and class n; according to the nature of the proposed investigation, one equation or the other will be the more suitable.

"The equation of a curve in point-coordinates, or as it may be termed the point-equation of the curve, is the relation which exists between the point-coordinates of any ineunt of the curve.

The equation of a curve in line-coordinates, or line-equation of the curve, is the relation which exists between the line-coordinates of any tangent of the curve."

(Cayley, *Collected Papers*, vol. ii., No. 158; 1859.)

For many purposes it is more convenient to consider the curve as on the one hand traced by a moving point, on the other enveloped by a moving line; the tracing point must then be regarded as moving along the tangent, the enveloping

line as rotating about the point of contact. This dual conception of the nature of a curve was first formulated by Plücker (*Theorie der Algebraischen Curven*, p. 200; 1839):—

"If a point move continuously along a straight line, while the straight line rotates continuously about the point, one and the same curve is enveloped by the line and described by the point."

56. Plainly the case $n=1$ is an exception to the statement just made regarding the dual aspect of a curve; a *point*, which is an envelope of class 1, cannot be regarded as furnishing a system of points, that is, it cannot be regarded as a locus; it has not a point equation; a *line*, which is a locus of order 1, cannot be regarded as furnishing a system of lines; that is, it cannot be regarded as an envelope; it has not a line equation.

57. Now an algebraic expression may be a product of factors; that is, the equation may split up into equations of lower degree; the curve considered under either aspect may be degenerate. A degenerate locus is composed of loci of lower order; thus a degenerate locus of the second order can only be two loci of the first order, and therefore a pair of straight lines. A degenerate envelope is composed of envelopes of lower class; thus a degenerate envelope of the second class can only be two envelopes of the first class, and therefore a pair of points.

Dual Interpretation of Algebraic Work.

58. The analytical discussion of any geometrical theorem consists of four parts:—

I. The statement of the geometrical data; there are certain elements, whose positions are controlled by given conditions:

II. The algebraic expression of these conditions, by which certain equations are obtained:

III. Algebraic combinations and transformations applied to these equations:

IV. Geometrical interpretation of the results of these algebraic operations.

In the purely algebraic parts of the discussion, II. and III., no attention is paid to the significance of the symbols; these may be point coordinates, or they may be line coordinates. But the whole proof of the theorem is in III. Hence in proving any theorem regarding points and lines we do *at the same time* prove another theorem regarding lines and

points; there is no question of deducing the one theorem from the other; the two are proved simultaneously.

It is in this dual interpretation of the algebraic work that the Principle of Duality is algebraically exhibited.

"This reciprocity" (between point and line in a plane) "can be formulated in the following manner; starting from the point, we obtain the straight line by joining two points; starting from the straight line, we obtain the point by the intersection of two lines. The simplest geometrical figure for the *point* is the straight line which it describes in moving along it; for the *straight line* it is the point which it envelopes in rotating about it.... The aggregate of these relations is expressed by the Principle of Duality; this principle asserts that certain theorems holding for point configurations can be transferred to line configurations; this reciprocity applies to the join of two points and the intersection of two lines, and extends to all constructions resulting from operations of this nature. It does not hold when other auxiliary means have to be employed, *e.g.* when a metric determination enters into the problem. But the principle is not thereby limited, for we shall find that all such metric relations can be expressed in terms of the others."

(Clebsch, *Vorlesungen über Geometrie*, t. i., p. 28; ed. Lindemann, 1876.)

EXAMPLES.

Point or line coordinates are to be used (or both together) according to the nature of the problem; and all results must be homogeneous.

1. Find the envelope of a line moving so that the product of its distances from two fixed points is constant.

Verify from à priori considerations that this envelope is of the second class.

2. Find the locus of a point moving so that the product of its distances from two fixed lines is constant.

3. Find the locus of a point moving so that the sum of its distances from n fixed lines is constant.

4. Find the envelope of a line moving so that the sum of its distances from n fixed points is constant.

5. Find the envelope of a line whose distance from a fixed point is constant.

6. Lines BP, CP are drawn to meet the sides of the triangle of reference in B', C'; if P move along a line through A, show that $B'C'$ passes through a fixed point on BC.

7. In Ex. 6, if P move along a fixed line $lx+my+nz=0$, find the line equation of the envelope of $B'C'$.

8. If $B'C'$ pass through a fixed point $f\xi+g\eta+h\zeta=0$, find the point equation of the locus of P.

9. A point moves along a fixed line; show that the envelope of its polar with regard to a triangle is of the second class.

10. A line passes through a fixed point; show that the locus of its pole with regard to a triangle is of the second order.

CHAPTER V.

DESCRIPTIVE PROPERTIES OF CURVES.

General Principles.

59. When homogeneous coordinates are employed, the investigation may proceed without any reference to the fundamental identical relations, the properties discussed being purely descriptive; on the other hand the properties being metric, the fundamental relations have to be taken into account. The general discussion of curves falls therefore naturally into two principal divisions; the second being subdivided according as the fundamental identical relation involved is that in point coordinates or that in line coordinates.

60. Eliminating z between an equation of degree n, $F(x, y, z)=0$, and a linear equation $fx+gy+hz=0$, the result is a homogeneous equation of degree n in x, y. It represents therefore n straight lines through the vertex C; from the way in which the equation was obtained, these lines pass through the intersections of the curve $F=0$ and the line. Hence the number of these intersections is n, and the curve $F=0$ is of order n.

Similarly the elimination of ζ between the line equation of a curve $\Phi=0$, and a linear equation (the equation of an arbitrarily chosen point) gives an equation homogeneous and of degree n in ξ, η. This represents n points on the line c, and these are the points in which c meets the tangents through the arbitrarily chosen point. There are therefore n such tangents, and the envelope $\Phi=0$ is of class n.

Note. The locus F has also a line equation whose degree, at this stage unknown, will be considered later; and the envelope Φ has a point equation, to be similarly determined (§ 68).

61. A principle that is of wide application is the follow-

ing:—If $f=0$, $f'=0$ be any two equations of the same degree, then $f+kf'=0$ is satisfied by all the common elements of $f=0$, $f'=0$; that is, if $f=0$, $f'=0$, be the point equations of two curves, then $f+kf'=0$ passes through all their common points; and if $\phi=0$, $\phi'=0$, be the line equations of two curves, then $\phi+k\phi'=0$ touches all their common tangents. The curves $f+kf'=0$, through the common points of $f=0$, $f'=0$, form a *pencil*; the curves $\phi+k\phi'=0$, touching the common tangents of $\phi=0$, $\phi'=0$, may conveniently be spoken of as a *range*. The curves forming either of these configurations, depending on one parameter k, form a singly infinite system; or, using the phraseology of § 4, the curve element of the configuration has one degree of freedom: the one parameter may be regarded as the one independent coordinate of the curve.

If now f, f', split up, this principle takes the form:—The curve $uu'+vv'=0$ (in point coordinates) passes through all points common to uu' and vv'; *i.e.* through all intersections of the pairs u, v; u, v'; u', v; u', v'; and in line coordinates, $uu'+vv'=0$ touches all the common tangents of the pairs u, v; u, v'; u', v; u', v'. As a special case, let u, v, w, w' be linear functions; then $uvs+ww't=0$ passes through (among others) the points uw, uw'. If now w' approach w indefinitely, the equation becomes $uvs+w^2t=0$; the points uw, uw' (W, W' in Fig. 20) approach one another indefinitely, and u becomes a tangent, whose point of contact is uw. Similarly the curve touches the line v at vw; the two lines u, v are tangents, and w is the chord of contact.

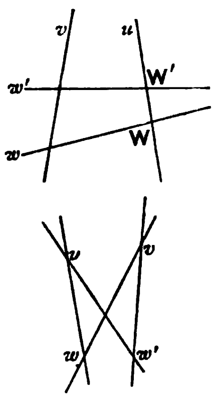

FIG. 20.

The interpretation of the equation $uvs+w^2t=0$ in line coordinates is that it represents a curve passing through the points u, v, the tangents at these points intersecting at the point w (Fig. 20).

The argument just used does not require u, v, w to be linear; the curve $us+w^2t=0$, in point coordinates, touches the curve $u=0$ at every point common to the curves $u=0$, $w=0$; and in line coordinates, this cuts the curve $u=0$ on every tangent common to $u=0$, $w=0$.

Again, no assumption having been made as to the nature of the coordinates used, they may be homogeneous or Cartesian.

Ex. 1. $$x^2-4y^2=36,$$
that is, $$(x-2y)(x+2y)=c^2.$$
This touches $x-2y=0$, $x+2y=0$, on the line $c=0$, *i.e.* at infinity. It is a hyperbola with $x-2y=0$, $x+2y=0$, for asymptotes.

Ex. 2. $$y^2=4x\text{ ; that is, } x\,.\,c=y^2.$$
This touches $x=0$, $c=0$, on the line $y=0$.

Ex. 3. $$xy=(x^2+y^2-a^2)^2.$$
This touches $x=0$ where it meets $x^2+y^2-a^2=0$, *i.e.* at the two points $x=0$, $y=a$; $x=0$, $y=-a$; and again it touches $y=0$ at $x=a$, $y=0$; $x=-a$, $y=0$.

These examples relate to Cartesian coordinates; examples in homogeneous coordinates, point and line, occur in the following pages.

Equations of the Second Degree, satisfying Three assigned Conditions.

62. We shall now consider equations of the second degree more especially, though not exclusively, since occasionally equations of higher degree afford better illustrations of the methods.

The general equation of the second degree in point coordinates is
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

This locus will pass through the vertex A of the triangle of reference if the coordinates of A, 1, 0, 0, satisfy the equation, that is, if $a=0$. Similarly it will pass through B, C if $b=0$, $c=0$. Hence the locus of the second order through the vertices of the triangle of reference is
$$fyz+gzx+hxy=0;$$
and the envelope of the second class touching the sides of the triangle of reference is
$$f\eta\zeta+g\zeta\xi+h\xi\eta=0.$$

The tangent at A can be determined by a direct method. Any line through A is $y=kz$. The lines joining B to the intersections of this line and the locus
$$fyz+gzx+hxy=0$$
are obtained by eliminating y; they are therefore
$$fkz^2+(g+hk)xz=0;$$
that is, $z=0$, which represents BA, and
$$fkz+(g+hk)x=0,$$
which represents BA' (Fig. 21). For $y=kz$ to be a tangent, the points A, A' must coincide, *i.e.* BA' must be the same

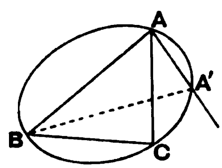

FIG. 21.

as BA; hence $g+hk=0$. The tangent at A is therefore

$$hy+gz=0.$$

Similarly the equation of the point of contact of BC with the envelope

$$f\eta\zeta+g\zeta\xi+h\xi\eta=0$$

is

$$h\eta+g\zeta=0.$$

Ex. 1. If three tangents be drawn to a locus of the second order, they meet the chords of contact in collinear points.

Ex. 2. If three points be taken on an envelope of the second class, the lines joining respectively one of these to the intersection of tangents at the other two are concurrent.

In the argument of this section x, y, z are simply any three lines; thus if U, V, W be a triangle other than the triangle of reference, having for sides the lines $u=0$, $v=0$, $w=0$, any locus of the second order through U, V, W, is

$$fvw+gwu+huv=0.$$

And even if u, v, w be functions not linear,

$$fvw+gwu+huv=0$$

represents a locus through the intersections of the pairs v, w; w, u; u, v; *e.g.* if u, v, w be of the second degree, this is a locus of the fourth order.

Ex. Find the general equation of a locus of the third order through A, B, C.

Here two of the three values given by $x=0$ are to be $y=0$, $z=0$; therefore the equation is of the form

$$xu+yzv=0;$$

where the whole expression being, by requirement, of degree 3, u is of the second degree and v of the first.

Similarly $y=0$ is to give $x=0$ or $z=0$; therefore u is of the form $sy+tz$, where s and t are linear. The three assigned conditions have now been satisfied, and the cubic locus has for its equation

$$yzv+zxt+xys=0,$$

where v, t, and s are any linear functions.

This result can also be obtained by taking the general cubic equation,

$$ax^3+\ldots+3a'x^2z+\ldots+3a''xz^2+\ldots+6dxyz=0,$$

and expressing that this is satisfied by the sets of values 1, 0, 0; 0, 1, 0; 0, 0, 1; it appears at once that $a=0$, $b=0$, $c=0$.

63. In the last section the configuration of the second degree was considered as having three assigned primary elements. It will now be considered as having three assigned secondary elements.

The conditions that the locus of the second order touch the sides of the triangle of reference are most simply determined from the consideration:—The line $x=0$ touches the locus if

the result of making $x=0$ in the equation be a perfect square; for this result gives the equation of the lines joining A to the two intersections. Hence the locus touches BC if

$$by^2+2fyz+cz^2=0$$

be a perfect square, that is, if $f^2=bc$; and it touches the other sides of the triangle of reference if $g^2=ca$, $h^2=ab$. Hence a, b, c must be all of the same sign, which may be taken positive. Writing therefore l^2, m^2, n^2 for a, b, c, the conditions found give $f=\pm mn$, $g=\pm nl$, $h=\pm lm$; and the inscribed locus is

$$l^2x^2+m^2y^2+n^2z^2\pm 2mnyz\pm 2nlzx\pm 2lmxy=0.$$

Now

$$l^2x^2+m^2y^2+n^2z^2+2mnyz+2nlzx+2lmxy=(lx+my+nz)^2,$$

and

$$l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx+2lmxy=(lx+my-nz)^2.$$

Hence the ambiguities must not be resolved in the ways here indicated; that is, they must not be taken $+$, $+$, $+$; $-$, $-$, $+$; they must be either $-$, $-$, $-$; or $-$, $+$, $+$.

If the second form be adopted, the equation becomes

$$l^2x^2+m^2y^2+n^2z^2-2mnyz+2nlzx+2lmxy=0,$$

and this, by means of the purely literal change involved in writing $-l$ for l, becomes

$$l^2x^2+m^2y^2+n^2z^2-2mnyz-2nlzx-2lmxy=0,$$

which may therefore be adopted as the standard form for the inscribed locus of the second order.

The equation can be reduced to the irrational form

$$(lx)^{\frac{1}{2}}+(my)^{\frac{1}{2}}+(nz)^{\frac{1}{2}}=0.$$

Similarly an envelope of the second class through the vertices of the triangle of reference has for its equation

$$(l\xi)^{\frac{1}{2}}+(m\eta)^{\frac{1}{2}}+(n\zeta)^{\frac{1}{2}}=0.$$

Ex. 1. If a locus of the second order be inscribed in a triangle, the lines joining the points of contact of the sides of the triangle to the vertices are concurrent.

Ex. 2. If an envelope of the second class be described about a triangle, the intersections of the tangents at the vertices and the opposite sides of the triangle are collinear.

Equation of the Derived Secondary Element.

64. The tangent at a point of a curve is defined as the line joining two consecutive points; and the point of contact of a tangent is the intersection of two consecutive lines. In either case the question is to find the equation of the

secondary element determined by two sets of coordinates x_1, y_1, z_1; $x_1+\delta x_1, y_1+\delta y_1, z_1+\delta z_1$; these being (1) point coordinates, (2) line coordinates.

If the equation be

$$lx+my+nz=0 \qquad\qquad (1),$$

l, m, n must satisfy

$$lx_1+my_1+nz_1=0 \qquad\qquad (2),$$

$$l(x_1+\delta x_1)+m(y_1+\delta y_1)+n(z_1+\delta z_1)=0,$$

from which, by subtraction,

$$l\delta x_1+m\delta y_1+n\delta z_1=0 \qquad\qquad (3).$$

From (2) and (3),

$$l:m:n=y_1\delta z_1-z_1\delta y_1 : z_1\delta x_1-x_1\delta z_1 : x_1\delta y_1-y_1\delta x_1.$$

If now $F=0$ be the homogeneous equation of the curve, and $F_1, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial y_1}, \frac{\partial F}{\partial z_1}$ be written for the values of $F, \frac{\partial F}{\partial x}$, etc., at the point x_1, y_1, z_1, we have, by Euler's theorem,

$$x_1\frac{\partial F}{\partial x_1}+y_1\frac{\partial F}{\partial y_1}+z_1\frac{\partial F}{\partial z_1}(=nF_1)=0;$$

and since $x_1+\delta x_1, y_1+\delta y_1, z_1+\delta z_1$ is on the curve,

$$\delta x_1\frac{\partial F}{\partial x_1}+\delta y_1\frac{\partial F}{\partial y_1}+\delta z_1\frac{\partial F}{\partial z_1}=0;$$

therefore

$$\frac{\partial F}{\partial x_1}:\frac{\partial F}{\partial y_1}:\frac{\partial F}{\partial z_1}=y_1\delta z_1-z_1\delta y_1 : z_1\delta x_1-x_1\delta z_1 : x_1\delta y_1-y_1\delta x_1.$$

Hence $$l:m:n=\frac{\partial F}{\partial x_1}:\frac{\partial F}{\partial y_1}:\frac{\partial F}{\partial z_1},$$

and equation (1) becomes

$$x\frac{\partial F}{\partial x_1}+y\frac{\partial F}{\partial y_1}+z\frac{\partial F}{\partial z_1}=0,$$

which is therefore the equation of the tangent to $F=0$ at the point x_1, y_1, z_1.

Similarly the equation of the point of contact with $\Phi=0$ of the line ξ_1, η_1, ζ_1, is

$$\xi\frac{\partial \Phi}{\partial \xi_1}+\eta\frac{\partial \Phi}{\partial \eta_1}+\zeta\frac{\partial \Phi}{\partial \zeta_1}=0.$$

65. Thus when the point equation of a curve, $F=0$, is known, the equation of the tangent at any point is known; the coordinates of the tangent are therefore known; and hence, theoretically, the relation to which these coordinates

are subject can be deduced; that is, the line equation of the curve can be found. This line equation is often called the tangential equation of the curve.

As an example of the process here outlined, consider the special locus of the second order

$$ax^2+by^2+cz^2=0.$$

The tangent at x_1, y_1, z_1 is

$$ax_1x+by_1y+cz_1z=0,$$

hence the coordinates of the tangent are $\xi=ax_1$, $\eta=by_1$, $\zeta=cz_1$. We have to find the relation satisfied by ξ, η, ζ.

Writing these equations in the form

$$x_1=\frac{\xi}{a}, \quad y_1=\frac{\eta}{b}, \quad z_1=\frac{\zeta}{c},$$

and making use of the relation expressing that x_1, y_1, z_1 is on the given locus, viz., $ax_1^2+by_1^2+cz_1^2=0$, the equation in ξ, η, ζ is found to be

$$\frac{\xi^2}{a}+\frac{\eta^2}{b}+\frac{\zeta^2}{c}=0.$$

In applying this process to the general curve $F=0$, it is necessary to eliminate x_1, y_1, z_1 from three equations of degree $n-1$, viz., $\frac{\partial F}{\partial x_1}=\xi$, $\frac{\partial F}{\partial y_1}=\eta$, $\frac{\partial F}{\partial z_1}=\zeta$, and the equation $F_1=0$. Now since

$$x_1\frac{\partial F}{\partial x_1}+y_1\frac{\partial F}{\partial y_1}+z_1\frac{\partial F}{\partial z_1}=nF_1,$$

the equation $F_1=0$ may be replaced by

$$\xi x_1+\eta y_1+\zeta z_1=0.$$

Even with this simplification the difficulty of the elimination in the general case is practically prohibitive. But if $n=2$, the equations are all linear, and the elimination can be at once performed. In this case

$$F=ax^2+by^2+cz^2+2fyz+2gzx+2hxy\,;$$

the equations are therefore

$$\left.\begin{aligned} ax_1+hy_1+gz_1&=\xi,\\ hx_1+by_1+fz_1&=\eta,\\ gx_1+fy_1+cz_1&=\zeta,\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Writing Δ for the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

these give,—*unless* $\Delta=0$, an exception whose significance appears in § 71,—

$$x_1\Delta=\begin{vmatrix} \xi & h & g \\ \eta & b & f \\ \zeta & f & c \end{vmatrix},\quad y_1\Delta=\begin{vmatrix} a & \xi & g \\ h & \eta & f \\ g & \zeta & c \end{vmatrix},\quad z_1\Delta=\begin{vmatrix} a & h & \xi \\ h & b & \eta \\ g & f & \zeta \end{vmatrix}.$$

Substituting these values in

$$\xi x_1+\eta y_1+\zeta z_1=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

the line equation of

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

is found to be

$$(bc-f^2)\xi^2+(ca-g^2)\eta^2+(ab-h^2)\zeta^2$$
$$+2(gh-af)\eta\zeta+2(hf-bg)\zeta\xi+2(fg-ch)\xi\eta=0,$$

that is, $A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta=0,$

where A, B, etc., are the minors of a, b, etc., in the determinant Δ.

And similarly the line equation being

$$a\xi^2+b\eta^2+c\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta=0,$$

the point equation is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0.$$

66. This result may be obtained in a slightly different form. Instead of solving equations (1) for x_1, y_1, z_1 and substituting in (2), we may simply eliminate x_1, y_1, z_1 from equations (1) and (2). The result is

$$\begin{vmatrix} a & h & g & \xi \\ h & b & f & \eta \\ g & f & c & \zeta \\ \xi & \eta & \zeta & \end{vmatrix}=0,$$

the same equation as before, but now expressed as a determinant.

This equation, in either of its forms, can be interpreted as a condition. Regard ξ, η, ζ as a known line, l, m, n; the condition that the line

$$lx+my+nz=0$$

touch the locus

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

is
$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & \end{vmatrix}=0.$$

67. The result of § 65 shows that (with the possible exception of the case $\Delta=0$) the general curve of the second order is also of the second class, and that the general curve of the second class is also of the second order. Any such curve is called a conic.

It is only for the special case $n=2$ that the general curve of order (or class) n has its order and class equal; therefore in speaking, for example, of a cubic, we ought properly to state whether we are referring to order or to class. We may speak of an order-cubic or a class-cubic; or the distinction may be drawn between point-cubic and line-cubic.

Formation of the Reciprocal Equation.

68. On account of the elimination required in the process of § 65 for forming the line equation, a different method is adopted if n be greater than 2. Forming the equation for the intersections of the curve $F=0$ and the line $\xi x+\eta y+\zeta z=0$ (by eliminating one of the variables) and expressing that two roots of this are to be equal, a condition is imposed on ξ, η, ζ; that is, the equation satisfied by ξ, η, ζ is found, and this is the desired line equation.

Ex. 1.
$$F=ax^2+by^2+cz^2=0,$$
$$\xi x+\eta y+\zeta z=0.$$

The equation for the intersections, obtained by the elimination of z is, when cleared of fractions by a multiplier ζ^2,
$$\zeta^2(ax^2+by^2)+c(\xi x+\eta y)^2=0,$$
that is,
$$(a\zeta^2+c\xi^2)x^2+2c\xi\eta xy+(b\zeta^2+c\eta^2)y^2=0.$$
The two roots of this are equal if
$$(a\zeta^2+c\xi^2)(b\zeta^2+c\eta^2)-c^2\xi^2\eta^2=0,$$
that is, if
$$\zeta^2(bc\xi^2+ca\eta^2+ab\zeta^2)=0.$$
The factor ζ^2, introduced as a multiplier in forming the equation for the intersections, is irrelevant; the line equation of the curve is therefore
$$bc\xi^2+ca\eta^2+ab\zeta^2=0,$$
that is,
$$\frac{\xi^2}{a}+\frac{\eta^2}{b}+\frac{\zeta^2}{c}=0.$$

Ex. 2.
$$F=x^3-y^2z=0,$$
$$\xi x+\eta y+\zeta z=0.$$
The equation for the intersections is
$$x^3\zeta+xy^2\xi+y^3\eta=0.$$
This is a cubic $f(\lambda)$ in $\lambda\left(=\frac{x}{y}\right)$; the condition for equal roots, obtained by the elimination of λ from $f(\lambda)=0$, $\frac{df}{d\lambda}=0$,

that is, from $$\zeta\lambda^3+\xi\lambda+\eta=0,$$
and $$3\zeta\lambda^2+\xi \quad =0,$$
is $$4\xi^3+27\eta^2\zeta =0.$$

Thus this particular order-cubic is also a class-cubic.

Ex. 3. $$F=x^3+y^3+z^3=0,$$
$$\xi x+\eta y+\zeta z=0.$$

The equation for the intersections is
$$(\xi^3-\zeta^3)x^3+3\xi^2\eta x^2y+3\xi\eta^2xy^2+(\eta^3-\zeta^3)y^3=0.$$

The condition that the cubic
$$a\lambda^3+3b\lambda^2+3c\lambda+d=0$$
have equal roots is
$$(bc-ad)^2=4(b^2-ac)(c^2-bd).$$

In the present example
$$bc-ad,\ b^2-ac,\ c^2-bd=\zeta^3(\xi^3+\eta^3-\zeta^3),\ \xi\eta^2\zeta^3,\ \xi^2\eta\zeta^3;$$
consequently the line $\xi x+\eta y+\zeta z=0$ is a tangent if
$$\zeta^6(\xi^3+\eta^3-\zeta^3)^2=4\xi^3\eta^3\zeta^6.$$
Plainly $\zeta=0$ does not make the line a tangent; the factor ζ^6, introduced in forming the equation, is irrelevant. The line equation of $x^3+y^3+z^3=0$ is therefore
$$(\xi^3+\eta^3-\zeta^3)^2=4\xi^3\eta^3,$$
that is, $$\xi^6+\eta^6+\zeta^6-2\eta^3\zeta^3-2\zeta^3\xi^3-2\xi^3\eta^3=0.$$

This order-cubic is therefore a class-sextic; and similarly the class-cubic
$$\xi^3+\eta^3+\zeta^3=0,$$
is the order-sextic
$$x^6+y^6+z^6-2y^3z^3-2z^3x^3-2x^3y^3=0.$$

If then the order m be 3, the class n may be as much as 6 or as little as 3. Thus the class does not depend solely on the order, and the order does not depend solely on the class.

69. If $F_m(x, y, z)=0$ be the point equation of a curve, the above process gives the line equation $\Phi_n(\xi, \eta, \zeta)=0$. Now corresponding to the locus $F_m(x, y, z)=0$ there is by the principle of duality an envelope $F_m(\xi, \eta, \zeta)=0$; and the point equation of this, found by the same algebraic work as before, is $\Phi_n(x, y, z)=0$. Instead of discussing the two equations
$$F_m(x, y, z)=0, \quad \Phi_n(\xi, \eta, \zeta)=0$$
for the one curve, it is often convenient to discuss the two distinct curves
$$F_m(x, y, z)=0, \quad \Phi_n(x, y, z)=0,$$
which are called *reciprocal curves.* Thus instead of attending simultaneously to the points and lines of the one curve, we consider the points of the two curves, knowing that all the line properties of $F_m(x, y, z)=0$ are exactly represented

by the point properties of $\Phi_n(x, y, z)=0$. These two reciprocal curves will now be spoken of as F and Φ.

Carrying the general comparison of Chapter IV. a step further, these two curves, F, Φ, are corresponding curves. To a double point on F, with tangents distinct or coincident, corresponds on Φ a double line, with points of contact distinct or coincident. For in each case the primary element presents itself twice in the same position, and we discriminate by means of the derived secondary elements. Now coincident tangents at a double point may be due to a cusp, which involves the turning back of the tracing point along the tangent; or to a tacnode, where there is contact of two distinct branches. In the latter case, the two branches have the same point and the same tangent; in the reciprocal there are therefore two branches with the same tangent and the same point of contact; that is, there is a tacnode on the reciprocal. At a cusp, the tracing point turns back along the tangent: that is, the primary element reverses the sense of its motion in the derived secondary element,

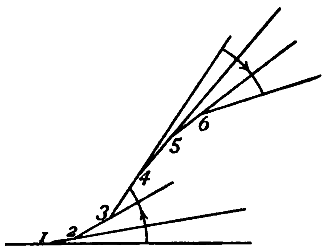

FIG. 22.

and therefore in the reciprocal the enveloping line changes the direction of its rotation about the point of contact (Fig. 22); it is an inflexional tangent; the point of contact, viz., the inflexion, corresponds to the cuspidal tangent. Thus the cusp and the inflexional tangent are properly stationary elements, while the node and the double tangent are simply double elements.

Ex. The line equation of

$$F=x^3+ky^2z=0 \text{ is } \xi^3+p\eta^2\zeta=0 \text{ (Ex. 2, § 68).}$$

Hence the reciprocal curve is

$$\Phi=x^3+py^2z=0.$$

The curve F passes through B, C; to see how it lies at C, let this be made the origin of Cartesian coordinates. Writing $ax+by+c$ for z (§ 32) the equation becomes $x^3+ky^2(ax+by+c)=0$, exhibiting a cusp at C (*i.e.* at 0, 0, 1), the cuspidal tangent being $y=0$ (*i.e.* the line 0, 1, 0). On the reciprocal Φ we are therefore to consider the line 0, 0, 1, and

the point 0, 1, 0; that is, the line $z=0$ and the point B. Making B the origin, the equation of the reciprocal Φ in Cartesian coordinates x, z is $x^3+pz(lx+mz+n)^2=0$, a form which shows that there is an inflexion at B, the line $z=0$ being the inflexional tangent. Thus to the cusp and cuspidal tangent on F there correspond on Φ the inflexional tangent and the inflexion.

Poles and Polars.

70. From any point x', y', z', n tangents can be drawn to a curve of class n. If the line equation of the curve be known, the coordinates of these n tangents can be determined (§ 60). If the point equation of the curve be given, the question is how to determine the equations of these n tangents. Let x_1, y_1, z_1 be the point of contact of any one; then the equation of the tangent is $x\frac{\partial F}{\partial x_1}+y\frac{\partial F}{\partial y_1}+z\frac{\partial F}{\partial z_1}=0$. The condition that this is to pass through x', y', z' shows that x_1, y_1, z_1 must satisfy the equation of degree $m-1$,

$$x'\frac{\partial F}{\partial x_1}+y'\frac{\partial F}{\partial y_1}+z'\frac{\partial F}{\partial z_1}=0\,;$$

and as x_1, y_1, z_1 is by hypothesis a point on the given curve, it is an intersection of the two curves

$$F=0 \qquad\qquad (1),$$

$$x'\frac{\partial F}{\partial x}+y'\frac{\partial F}{\partial y}+z'\frac{\partial F}{\partial z}=0 \qquad\qquad (2).$$

This derived curve (2) is called the first polar of x', y', z' with regard to the given curve F.

But if there be a point at which $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$, at this point equation (2) is satisfied, and (1) is also satisfied, for

$$x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}+z\frac{\partial F}{\partial z}=mF,$$

and therefore $F=0$. But the equation for the tangent is now illusory; and therefore in determining the points of contact of tangents from x', y', z' by means of the intersections of (1) and (2), these exceptional points must be excluded. The result must therefore be stated:—

The intersections of a curve and the first polar of a point with regard to the curve give (1) *the points of contact of all tangents from the point;* (2) *all points, if such there be, at which* $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ *vanish simultaneously.*

Note. Two equations of degrees m, p, homogeneous in three variables, have mp sets of roots; this is proved in works on algebra (see Salmon's *Higher Algebra*, § 73); hence two curves of orders m, p have mp common points; and since the class of a curve of order m cannot exceed the number of intersections of two curves of orders m, $m-1$, $n \not> m(m-1)$; and reciprocally, $m \not> n(n-1)$.

The expression for the number of intersections of two curves will however not be assumed, it is simply noted here for purposes of illustration; when the number of intersections of two curves is required in an argument, an independent proof is given.

71. For the special case of the conic, the first polar—or more simply, the polar—is a straight line. This meets the conic in two points; before it can be asserted that these are points of contact of tangents from x', y', z', the possibility of the simultaneous vanishing of $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ must be examined. The question is, can the three equations

$$ax+hy+gz=0,$$
$$hx+by+fz=0,$$
$$gx+fy+cz=0,$$

be simultaneously satisfied? Not unless $\Delta=0$, where Δ is the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

This is the possibility that presented itself in § 65, and the result of that section agrees with the result now obtained, viz., the curve of the second order is also of the second class, unless $\Delta=0$. The significance of this condition appears from the fact that the polar of any point with regard to a line-pair passes through the intersection of the lines, this general concurrence of the polars being moreover a sufficient condition for the degeneration of the locus of the second order. For the polar of x', y', z' with regard to $yz=0$ is $yz'+zy'=0$, which passes through the point yz; and if all polars pass through a point, take this as yz; then the values 1, 0, 0 for x, y, z make the linear expressions $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ vanish; consequently no one of these contains x, and so the expression F does not contain x, but is homogeneous of the second degree in y, z, and is therefore the product of linear factors. Hence the condition $\Delta=0$, which primarily expresses the concurrence of all polars, is the condition that the linear expression

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

split up into factors; it is the condition that the locus of the

second order degenerate to a line-pair, or that the envelope of the second class degenerate to a point-pair.

Since a line, or a pair of lines, cannot be regarded as an envelope, and reciprocally the point, or the pair of points, cannot be regarded as a locus, it is right and to be expected that the ordinary process for finding the reciprocal equation should, in this case, prove illusory.

72. Dealing with a proper conic, there are two tangents from any point. The construction for the polar of a given point (pole) is therefore:—Draw the two tangents, mark their points of contact, the join of these is the polar. The reciprocal construction starts from a line and leads to a point:—Mark the two intersections of the line with the conic, draw the tangents at these, their intersection gives the point. Comparing these, the line and point in the second construction are seen to be polar and pole; the two diagrams are the same; each is its own reciprocal, for pole and polar are reciprocal.

Since the tangents or intersections may be conjugate imaginaries, these constructions are not generally available; others will be given later. (See § 78).

73. The polar of x_1, y_1, z_1 is

$$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0;$$

this is symmetrical in the two sets x_1, y_1, z_1; x, y, z. Hence *if the polar of P pass through Q, the polar of Q passes through P;* and similarly *if the pole of p lie on q, the pole of q lies on p.* Pairs of elements related in this way are said to be conjugate. (See § 78.) Hence a point has an indefinite number of conjugates, all lying on the polar; and a line has an indefinite number of conjugates, all passing through the pole. The pole and polar are also spoken of as conjugate; taking any figure composed of points and lines, the polars and poles of these with reference to any fundamental conic form the conjugate figure; thus a triangle gives a conjugate triangle; a complete quadrilateral gives as conjugate a complete quadrangle.

If the fundamental conic be the imaginary conic $x^2+y^2+z^2=0$, the point f, g, h has for its conjugate the line $fx+gy+hz=0$; that is, the point whose coordinates are f, g, h is conjugate to the line whose coordinates are f, g, h. This introduces a geometrical connection between the two theories which have hitherto been regarded as independent parallel theories; the two explanations of any algebraic work are *conjugate*, or, to use the more general

term, they are *reciprocal*, and the two figures are reciprocal with regard to the fundamental conic. This view is discussed in the sections on Reciprocation (§§ 249, etc.), for the present the two theories will still be treated as independent.

74. The equation of the pair of tangents from $P(x', y', z')$ to the conic $F=0$ can be expressed by means of the equation of the polar of P,

$$p=x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz')=0.$$

Any conic touching F at the two points where it is met by the line p, that is, any conic having double contact with F on the line p, is

$$F+kp^2=0.$$

The conic to be found is a line-pair; hence in any particular example the value of k can be determined. The general value of k is more simply obtained by expressing that this conic passes through P. This is a sufficient condition; for, calling the two tangents u, v, and their points of contact U, V, the conic $F+kp^2=0$ meets u in two points at U; if it meet u also in P, this gives three points on u, and therefore u must form a part of the conic. Hence k is given by the equation

$$F'+kp'^2=0.$$

But the form of the expressions F, p, shows that p' and F' are identically the same, consequently the equation of the pair of tangents is

$$FF'-p^2=0.$$

Similarly if $\Phi=0$ be the line equation of a conic, and $\varpi=0$ the equation of the pole of the line p (ξ', η', ζ'), any conic having double contact with Φ, ϖ being the pole of the chord of contact, is

$$\Phi+k\varpi^2=0;$$

and the equation of the intersections of ξ', η', ζ' and Φ is

$$\Phi\Phi'-\varpi^2=0.$$

Examples.

1. Show that the lines l_1, m_1, n_1; l_2, m_2, n_2 are conjugate with regard to

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

if

$$(bc-f^2)l_1l_2+(ca-g^2)m_1m_2+(ab-h^2)n_1n_2+(gh-af)(m_1n_2+m_2n_1)$$
$$+(hf-bg)(n_1l_2+n_2l_1)+(fg-ch)(l_1m_2+l_2m_1)=0.$$

2. Find the coordinates of the point of intersection of a line-pair given by a point equation of the second degree.
Find also the equation of this point.

3. State the results reciprocal to those found in Ex. 2.

4. Apply the condition $\Delta=0$ to show that

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C=0$$

represents a pair of points; and show that the equation of the line joining these points is

$$ax+by+cz=0.$$

5. Let u, v, v' be three lines; denote the points vv', uv, uv' by ρ, ϖ, ϖ'. Show that the line equation of

$$u^2=kvv'$$

is of the form

$$\rho^2=\lambda\varpi\varpi'.$$

6. Find the reciprocal to $x^m y^n=z^{m+n}$.

7. If the conic reduce to a line-pair, show that any point in the plane is conjugate to the intersection of the lines.

8. With respect to a line-pair every point has a polar, but not every line has a pole.

9. Discuss the results for a point-pair analogous to those given in Ex. 7 and Ex. 8 for a line-pair.

Conics with Four assigned Elements.

75. The equation of a conic through three points, or touching three lines, left two constants to be determined, viz. (§ 62), $f:g:h$. The equation of a conic through four points, or touching four lines, contains therefore one undetermined quantity.

Let the four points be the intersections of lines $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, these lines being taken in the order given, so that the four points are $P(\alpha\beta)$, $Q(\beta\gamma)$, $R(\gamma\delta)$, $S(\delta\alpha)$. One conic through $PQRS$ is $\alpha\gamma=0$; another is $\beta\delta=0$. The conic $\alpha\gamma=k\beta\delta$ passes through all the intersections of $\alpha\gamma$, $\beta\delta$, that is, through P, Q, R, S; and since this equation contains one undetermined quantity, it is the most general equation of a conic through the four given points. Similarly, if α, β, γ, δ be linear in ξ, η, ζ, so that $\alpha=0$, etc. represent points, the line equation of the conic touching $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$, is $\alpha\gamma=k\beta\delta$.

76. The coordinates of the four elements were here supposed given by means of equations; but the elements themselves being given, and the choice of coordinates left unrestricted, the equation of the conic can be found in a useful form.

The four given elements determine one of the configurations considered in §§ 43, 44; taking the diagonal triangle for triangle of reference, and choosing coordinates properly, the four elements have coordinates 1, ± 1, ± 1. The general equation of the second degree is satisfied by

$$\begin{array}{rrrl} 1, & 1, & 1 & \text{if } a+b+c+2f+2g+2h=0\ldots\ldots\ldots(1), \\ -1, & 1, & 1 & \text{if } a+b+c+2f-2g-2h=0\ldots\ldots\ldots(2), \\ 1, & -1, & 1 & \text{if } a+b+c-2f+2g-2h=0\ldots\ldots\ldots(3), \\ 1, & 1, & -1 & \text{if } a+b+c-2f-2g+2h=0\ldots\ldots\ldots(4). \end{array}$$

Now (1) and (2) give $\quad g+h=0,$

(3) and (4) give $\quad g-h=0;$

hence $g=0$, $h=0$, and similarly $f=0$.

The point equation of the conic through the four points 1, ± 1, ± 1 is therefore

$$ax^2+by^2+cz^2=0,$$

with the condition $a+b+c=0$; and the line equation of the conic touching the four lines 1, ± 1, ± 1 is

$$a\xi^2+b\eta^2+c\zeta^2=0,$$

with the condition $a+b+c=0$.

Note. If the elements be taken as p, $\pm q$, $\pm r$, the equations obtained are of the same form, with the condition $ap^2+bq^2+cr^2=0$.

Now the point equation of $a\xi^2+b\eta^2+c\zeta^2=0$ is

$$bcx^2+cay^2+abz^2=0;$$

and the lines whose coordinates are 1, ± 1, ± 1 have equations $x\pm y\pm z=0$; consequently the point equation of the conic touching the four lines $x\pm y\pm z=0$ is

$$bcx^2+cay^2+abz^2=0,$$

with the condition $a+b+c=0$; or it may be written

$$Ax^2+By^2+Cz^2=0,$$

with the condition $\dfrac{1}{A}+\dfrac{1}{B}+\dfrac{1}{C}=0.$

Ex. Two conics being drawn through four points, prove that their eight tangents at these points all touch one conic.

Choosing coordinates as above, the conics are

$$px^2+qy^2+rz^2=0\ldots\ldots\ldots\ldots(1), \qquad p'x^2+q'y^2+r'z^2=0\ldots\ldots\ldots\ldots(2),$$

with the conditions

$$p+q+r=0, \quad p'+q'+r'=0 \quad\ldots\ldots\ldots\ldots\ \ldots\ldots\ldots\ldots(3).$$

The tangent to (1) at 1, 1, 1 is $px+qy+rz=0$; and similar equations are found for the other tangents. Hence the eight lines to be considered have coordinates

$$p,\ \pm q,\ \pm r; \quad p',\ \pm q',\ \pm r';$$

and it is to be shown that these satisfy a quadratic relation

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta = 0.$$

The first set of four will satisfy this if

$$f=0, \quad g=0, \quad h=0 \qquad \qquad (4),$$

$$ap^2+bq^2+cr^2=0 \qquad \qquad (5);$$

the second set will satisfy this if in addition

$$ap'^2+bq'^2+cr'^2=0 \qquad \qquad (6).$$

Hence a, b, c must be determined from the linear equations (5), (6); these give

$$a, b, c = q^2r'^2 - q'^2r^2, \; r^2p'^2 - r'^2p^2, \; p^2q'^2 - p'^2q^2;$$

but the conditions (3) give

$$qr' - q'r = rp' - r'p = pq' - p'q;$$

hence $\qquad a, b, c = qr' + q'r, \; rp' + r'p, \; pq' + p'q,$

and the line equation of the conic touched by all eight lines is

$$(qr' + q'r)\xi^2 + (rp' + r'p)\eta^2 + (pq' + p'q)\zeta^2 = 0.$$

Ex. 1. A conic can be drawn to touch the six lines that join the vertices of a triangle to the points in which the opposite sides are cut by any conic.

Ex. 2. A conic can be drawn through the six points in which the sides of a triangle are met by the tangents drawn from the opposite vertices to any conic.

77. The equation of a conic through four points $1, \pm 1, \pm 1$, being $ax^2+by^2+cz^2=0$, with the condition $a+b+c=0$, there is one undetermined constant in the equation, and this parameter is involved linearly; the conics form that particular singly infinite system defined as a pencil. For one special conic of the system, obtained by taking $a=0$, is $y^2-z^2=0$, and another is $x^2-z^2=0$; and the general equation of the system

$$ax^2+by^2+cz^2=0$$

can be written

$$a(x^2-z^2)+b(y^2-z^2)+(a+b+c)z^2=0,$$

which, by the condition $a+b+c=0$, is reduced to

$$a(x^2-z^2)+b(y^2-z^2)=0,$$

that is, to $u+\lambda v=0$, the typical form for a pencil of curves. Similarly the conics touching four given lines are represented by $\phi+\lambda\psi=0$, and therefore form a range.

78. The equation

$$ax^2+by^2+cz^2=0,$$

written in the form

$$(\sqrt{a}x+\sqrt{-b}y)(\sqrt{a}x-\sqrt{-b}y)=(\sqrt{-c}z)^2,$$

shows that $\sqrt{a}x\pm\sqrt{-b}y=0$ are tangents, $z=0$ being the chord of contact. Hence each vertex of the triangle of reference is the pole of the opposite side; that is, $a, b, c,$

A, B, C are conjugate to A, B, C, a, b, c; the triangle is its own conjugate, and is said to be self-conjugate with regard to the conic.

For the conic to be real, the coefficients a, b, c must be not all of the same sign; let c be negative; write l^2, m^2, $-n^2$ for a, b, c, so that the equation can be written in the forms

$$l^2x^2+m^2y^2=n^2z^2,$$
$$n^2z^2-l^2x^2=m^2y^2,$$
$$n^2z^2-m^2y^2=l^2x^2,$$

showing that the tangents from C are imaginary, while those from A, B are real. Thus for a real self-conjugate triangle, the vertices are one inside, two outside the conic; and the sides cut the conic, one in imaginary points, two in real points.

From the properties of the triangle ABC, which was chosen by means of the four points P, Q, R, S, a construction for the polar of a point A is deduced. Draw through A any two chords, PQ, RS; join the four points so determined in the two remaining possible ways, so determining points B, C; BC is the polar of A (Fig. 23). This construction applies whether A is inside or outside the conic.

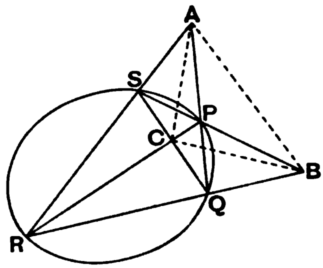

FIG. 23.

It should be noticed in Fig. 23 that taking different positions of the chords APQ, ASR, different positions can be obtained for B, C, but all lie on the line there determined as BC.

The construction here given shows that the join of conjugate points is cut harmonically by the conic. For let A, A', be conjugates; take AA' for one chord through A, e.g. APQ; then since A' is conjugate to A, the polar BC passes through A'; hence by the harmonic properties of the figure, A, A' are harmonic with respect to P, Q, and therefore with respect to the conic. It is on account of this property that the pairs of elements are said to be

conjugate with respect to the conic; they are *harmonic conjugates.* Similarly two conjugate lines, AB, AC are harmonic conjugates with regard to the tangents from their intersection A. The two points, two lines, or point and line might therefore with advantage be called harmonic instead of conjugate.

To construct a self-conjugate triangle, that is, a *harmonic triangle*, any point A is chosen at random; B is then chosen as any point on the polar of A, and the triangle is thereby completely determined. Hence all the triangles that are self-conjugate with regard to a given conic form a three-fold infinity.

Note. In general, the conic having the triangle uvw as a self-conjugate triangle is

$$lu^2+mv^2+nw^2=0;$$

hence the conics with regard to which a given triangle is self-conjugate form a two-fold infinity.

79. That any two conics intersect in four points, real or imaginary, appears from the fact that the elimination of z between the two equations leads to an equation of degree 4 in $x:y$. Similarly any two have four common tangents. From the four common points, or from the four common tangents, a triangle can be constructed self-conjugate with regard to each of the conics.

Ex. Show that the two constructions lead to the same triangle.

The four common points being all real, or all imaginary, the self-conjugate triangle is real (§ 51). Taking it as triangle of reference, the two conics are

$$ax^2+by^2+cz^2=0,$$
$$a'x^2+b'y^2+c'z^2=0.$$

The line equations of these conics are

$$bc\xi^2+ca\eta^2+ab\zeta^2=0,$$
$$b'c'\xi^2+c'a'\eta^2+a'b'\zeta^2=0;$$

consequently the coordinates of the common tangents, determined by solving these equations, are given by

$$\xi^2:\eta^2:\zeta^2=aa'(bc'-b'c):bb'(ca'-c'a):cc'(ab'-a'b).$$

If all the expressions on the right have the same sign, the four values of ξ, η, ζ are real; if the expressions have not all the same sign, the sets ξ, η, ζ are imaginary. Hence all four intersections being of the same nature (all real or all imaginary), the four common tangents are all real or all imaginary; and reciprocally, all four common tangents being

of the same nature, the four common points are all real or all imaginary.

The only cases omitted here are *one*, for the nature of the points, and *one*, for the nature of the lines. These must therefore go together; that is, the intersections being two real and two imaginary, the common tangents are two real and two imaginary. This case is of no special interest; but the cases where the four fundamental elements are all real or all imaginary require more detailed investigation.

80. Let there be four points, all real; their coordinates may therefore be taken as $1, \pm 1, \pm 1$. Any conic of the pencil is

$$ax^2+by^2+cz^2=0,$$

with the condition $$a+b+c=0.$$

Hence one of the three coefficients has the opposite sign to the other two; the real conics of the pencil fall into three sets, for which the signs of the coefficients are

$$\begin{array}{ll} \text{(i.)} & a-,\quad b \text{ and } c+, \\ \text{(ii.)} & b-,\quad c \text{ and } a+, \\ \text{(iii.)} & c-,\quad a \text{ and } b+; \end{array}$$

these three sets are geometrically distinguished by being met in imaginary points by the lines x, y, z, respectively.

The coordinates of the common tangents to two conics of the pencil are given by

$$\xi^2 : \eta^2 : \zeta^2 = aa'(bc'-b'c) : bb'(ca'-c'a) : cc'(ab'-a'b).$$

I. Let the two conics belong to the same set, for example the third, so that c, c' are negative.

There is no loss of generality in choosing numerical values for c, c'; take therefore

$$c=-1, \text{ whence } a+b=1;$$
$$c'=-1, \text{ whence } a'+b'=1.$$

Since a, b, a', b' and the product cc' are all positive, the signs of the expressions on the right are the same as those of

$$-b+b', \qquad -a'+a, \quad ab'-a'b,$$

that is, of $$a-1+1-a', \quad -a'+a, \quad a(1-a')-a'(1-a),$$

and therefore of $$a-a', \qquad a-a', \quad a-a'.$$

These are all of the same sign, and consequently ξ, η, ζ, are real; that is, members of the same set have real common tangents.

II. Let the two conics belong to different sets, for example, the third and second; a, b are now positive, and c is negative; a', c' are positive, b' is negative.

Take $\quad c=-1$, whence $a+b=1$;
$c'=+1$, whence $a'+b'=-1$.

The signs now depend on those of

$$b+b',\ -(-a'-a),\quad -(ab'-a'b),$$

that is, of $\quad -(a+a'),\quad +(a+a'),\quad -(ab'-a'b)$.

Here there is a difference in sign, and the coordinates ξ, η, ζ are imaginary; members of different sets have imaginary common tangents.

81. Now let the four points be imaginary; § 52 shows that their coordinates may be taken as i, ± 1, ± 1. Hence in the equation

$$ax^2+by^2+cz^2=0,$$

the coefficients are subject to the condition

$$-a+b+c=0.$$

Just as before, for a real conic, a, b, c have not all the same sign; there are now only two sets to consider, viz.,

(i.) a and $b+$, $c-$,
(ii.) a and $c+$, $b-$.

The three expressions whose signs are to be compared are

$$aa'(bc'-b'c),\quad bb'(ca'-c'a),\quad cc'(ab'-a'b);$$

that is,

$$aa'(bc'-b'c),\quad bb'\big(c(b'+c')-c'(b+c)\big),\quad cc'\big(b'(b+c)-b(b'+c')\big);$$

that is,

$$aa'(bc'-b'c),\ -bb'(bc'-b'c),\quad -cc'(bc'-b'c),$$

whose signs are the same as those of

$$-aa',\quad bb',\quad cc'.$$

I. Let the conics belong to the same set; then these are $-$, $+$, $+$, and the common tangents are imaginary.

II. Let the conics belong to different sets; these are $-$, $-$, $-$, and the common tangents are real.

82. Hence for a pencil through real points, the real conics fall into three sets, every one of a set having real tangents in common with any other of its set, no two from different sets having real common tangents (Fig. 24 (a)). The three sets are characterized by being met in imaginary points by BC, CA, AB, respectively.

For a pencil through imaginary points, the real conics fall into two sets, no one of a set having real tangents in common with any other of its set, but any one of a set having real tangents in common with any one of the other set (Fig. 24 (b)).

The two sets are characterized by being met in imaginary points by AB, AC, respectively.

Reciprocally, for a range with real tangents, the real conics fall into three sets, every one of a set visibly intersecting

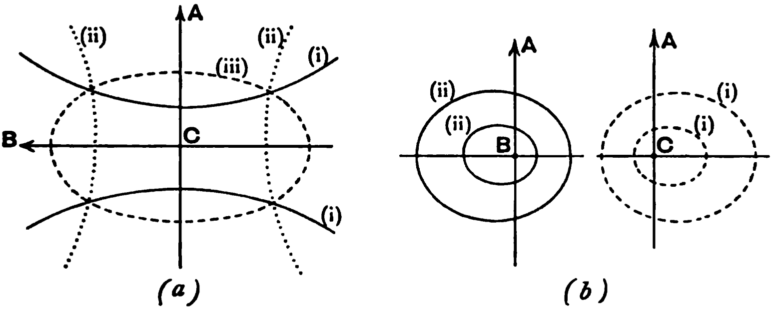

FIG. 24.

every other of the set, no two from different sets visibly intersecting (Fig. 25 (a)). The three sets are characterized by having imaginary tangents from A, B, C; that is, by being met in imaginary points by BC, CA, AB, respectively.

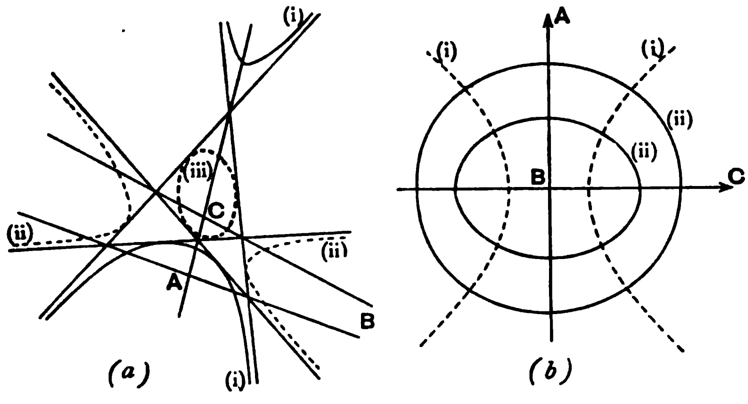

FIG. 25.

For a range with imaginary tangents, the real conics fall into two sets, no one of a set visibly intersecting any other of that set, but any one of a set visibly intersecting any one of the other set (Fig. 25 (b)). The two sets are characterized by having imaginary tangents from C, B; that is, by being met in imaginary points by AB, AC, respectively.

Note. In Fig. 24 (a) and in Fig. 25 (a) one conic is shown in every set; in Fig. 24 (b) two in each set, and in Fig. 25 (b) one in set (i.), two in set (ii.) are shown.

Applying to the case of conics a term lately suggested* to express the appearance of certain configurations, two conics with imaginary common points and imaginary common tangents (or, we may say, two conics that neither intersect nor connect visibly), are said to be *nested*; either set in Fig. 24 (*b*) or in Fig. 25 (*b*) is a *nest*. Thus if two pairs of imaginary elements be given for the determination of a system of conics, the real conics fall into two nests, which visibly connect or intersect according as the given imaginary elements are points or lines.

EXAMPLES.

1. Show that there is one point conjugate to any given point with respect to each of two conics; but that if the given point have one of three particular positions, the conjugate becomes indeterminate, being any point on a certain straight line.

2. Show that points conjugate with respect to each of two conics are conjugate with respect to all conics of the pencil.

3. Find the locus of the pole of a fixed line, and the envelope of the polar of a fixed point, with respect to (i.) a pencil, (ii.) a range, of conics.

4. The points P, Q are conjugate with respect to a pencil of conics; show that if P describe a straight line, Q describes a conic; and that this conic is the locus of the pole of the given line with respect to the pencil. How is it situated with regard to the pencil? (Poncelet.)

5. Find the envelope of the tangents at the points where a fixed transversal meets a pencil of conics.

6. Find the locus of the points of contact of tangents from a fixed point to a range of conics.

7. The polars of a point with regard to four conics of a pencil form a pencil whose cross-ratio is independent of the position of the point.

Note. This cross-ratio is called the cross-ratio of the conics; it may be determined as the cross-ratio of the four tangents at any one of the common points. (Chasles.)

8. Show that three conics of a pencil are line-pairs, and that of these three certainly one is real. Discuss the reciprocal idea.

9. Can a pencil of conics ever be a range?

* See Mr. Hulburt's account of the paper by Hilbert (*Math. Annalen*, t. xxxviii.; 1891) in the *Bulletin of the New York Mathematical Society*, vol. i. p. 197.

10. Taking any three conics, the polars of a point P are not ordinarily concurrent; find the condition to be satisfied by P in order that these polars may be concurrent; and apply this to prove that pairs of points harmonic with respect to three conics lie on a curve of order three.

11. Show that points harmonic with respect to the three conics $u=0$, $v=0$, $w=0$, are harmonic with respect to every conic of the *net*

$$lu+mv+nw=0.$$

12. If the line equations of the two conics $U=0$, $V=0$ be $\Phi=0$, $\Psi=0$, the line equation of the pencil $U+kV=0$ is $\Phi+k\Sigma+k^2\Psi=0$; and the point equation of the range $\Phi+\lambda\Psi=0$ is $U+\lambda S+\lambda^2 V=0$, where S, Σ, are expressions of the second degree.

Show that the line equation of the pencil represents a range if $\Sigma\equiv A\Phi+B\Psi$. Hence find the conditions that the conics

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$
$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0,$$

may have double contact.

On the Number of Conditions determining a Conic.

83. The general equation of the second degree contains six terms, and therefore it involves five disposable constants, viz., the ratios of the six coefficients; hence a conic, whether regarded as a locus or as an envelope, is determined by five conditions. Thus the three conditions of passing through A, B, C, or touching a, b, c, left *two* constants to be determined; the four conditions of passing through four points, or touching four lines, left *one*. Similarly the general equation of degree m contains $\frac{1}{2}(m+1)(m+2)$ terms, the number of disposable constants is therefore

$$\tfrac{1}{2}(m+1)(m+2)-1,$$

i.e. $\frac{1}{2}m(m+3)$; the general curve of order m, or of class m, is determined by $\frac{1}{2}m(m+3)$ independent conditions.

Considering now the conic, determined by five conditions, let these be simply that certain point or line elements are given.

I. *Given five points, or five lines.*

The coefficients have to satisfy five equations such as

$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0;$$

that is, they are given by five linear equations, and are therefore determined uniquely.

Note. It should be observed that k linear equations, homogeneous in $k+1$ quantities, cannot be inconsistent, for a reason that a single example will make clear.

Equations such as $3x+2y+1=0$, $6x+4y-3=0$, are called inconsistent in some books on elementary algebra, on account of the result obtained by the ordinary method of solution. But writing them in the homogeneous form,

$$3x+2y+z=0, \quad 6x+4y-3z=0,$$

the solution is seen to be $z=0$, $x:y=2:-3$; hence the equations are not inconsistent. Compare the discussion of parallel lines in § 25.

Thus the locus of the second order is determined uniquely by five points. Now the general locus of the second order is also the general envelope of the second class, this is therefore determined uniquely by five points; and by the principle of duality five lines determine the conic uniquely whether it be regarded as an envelope or as a locus. Hence one conic can be drawn to pass through five points, or to touch five lines.

II. *Given four points and one line, or four lines and one point.*

Take the diagonal triangle of the four elements as triangle of reference, and choose coordinates so that the four elements are 1, ±1, ±1. Then the equation is

$$ax^2+by^2+cz^2=0,$$

with the condition

$$a+b+c=0 \quad\text{..........................}(1),$$

and the condition of contact with the given line

$$px+qy+rz=0.$$

This last requires

$$bcp^2+caq^2+abr^2=0 \quad\text{..........................}(2):$$

hence a, b, c must be determined from equations (1) and (2); these give two sets of values for $a:b:c$; consequently two conics can be drawn to pass through four points and touch one line, or to touch four lines and pass through one point. Thus the five conditions determine the conic, though they determine it as one of two.

Ex. To what condition must the line be subject in order that the two conics may coincide?

III. *Given three points and two lines, or three lines and two points.*

Let the triangle of reference be the one determined by the three given elements, then the conic is

$$fyz+gzx+hxy=0.$$

From the conditions of contact with the lines

$$px+qy+rz=0, \quad p'x+q'y+r'z=0,$$

$$f^2p^2+g^2q^2+h^2r^2-2ghqr-2hfrp-2fgpq=0 \quad \ldots\ldots(1),$$

$$f^2p'^2+g^2q'^2+h^2r'^2-2ghq'r'-2hfr'p'-2fgp'q'=0 \quad \ldots\ldots(2).$$

Equations (1) and (2) give four sets of values $f:g:h$. Four conics can therefore be drawn to pass through three points and touch two lines, or to touch three lines and pass through two points. Here again the five conditions determine the conic, but they determine it as one of four.

An assigned condition may be such as to impose more than one relation on the coefficients; it is then counted as equivalent to a certain number of simple conditions.

EXAMPLES.

1. Show that to be given one pair of conjugate points (or lines) amounts to one condition. Is the condition linear or quadratic in the coefficients?

How many pairs of conjugates determine a conic?

2. Show that to be given a pole and polar amounts to two conditions. How many poles and polars determine a conic?

3. ABC is self-conjugate with regard to a conic; how many conditions are here imposed on the conic?

4. A conic is degenerate; how many conditions are here imposed?

Condition that Six Elements may belong to a Conic.

84. Since a conic, quà locus or envelope, is determined by five points or lines, any sixth point or line will not belong to the conic unless certain conditions are complied with. The essential dependence of any sixth element on the determining five is expressed by various theorems:—Pascal's, with the reciprocal, Brianchon's; Chasles', which depends on cross-ratio; and Desargues', which is most conveniently stated by means of the conception of Involution, and is therefore given later (§ 180).

85. *Pascal's Theorem. The intersections of opposite sides of a hexagon in a conic are collinear.*

Brianchon's Theorem. The joins of opposite vertices of a hexagon about a conic are concurrent.

In the two theorems the six given elements, which by hypothesis belong to a conic, may be combined in any order to form the hexagon: there are therefore 60 different hexagons, to

every one of which the theorem applies; consequently six points on a conic give 60 Pascal lines; and six tangents to a conic give 60 Brianchon points.

The theorems can be proved by a direct use of coordinates, point coordinates for Pascal's theorem, line coordinates for Brianchon's theorem.

Take for triangle of reference that determined by the alternate elements 1, 2, 3 (Fig. 26); let the opposite elements 1′, 2′, 3′ have coordinates x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3.

The equations of 23′, 2′3 are

$$\frac{x}{x_3}=\frac{z}{z_3},\quad \frac{x}{x_2}=\frac{y}{y_2};$$

therefore the derived element I. is $1,\ \dfrac{y_2}{x_2},\ \dfrac{z_3}{x_3}$;

the derived element II. is $\dfrac{x_1}{y_1},\ 1,\ \dfrac{z_3}{y_3}$;

the derived element III. is $\dfrac{x_1}{z_1},\ \dfrac{y_2}{z_2},\ 1.$

The determinant D formed with these coordinates is

$$\begin{vmatrix} 1 & \dfrac{y_2}{x_2} & \dfrac{z_3}{x_3} \\ \dfrac{x_1}{y_1} & 1 & \dfrac{z_3}{y_3} \\ \dfrac{x_1}{z_1} & \dfrac{y_2}{z_2} & 1 \end{vmatrix}, \;=x_1y_2z_3\begin{vmatrix} \dfrac{1}{x_1} & \dfrac{1}{y_1} & \dfrac{1}{z_1} \\ \dfrac{1}{x_2} & \dfrac{1}{y_2} & \dfrac{1}{z_2} \\ \dfrac{1}{x_3} & \dfrac{1}{y_3} & \dfrac{1}{z_3} \end{vmatrix}.$$

Now by hypothesis the three elements 1′, 2′, 3′ belong to a conic 1, 2, 3: consequently a relation

$$lyz+mzx+nxy=0,$$

that is,

$$\frac{l}{x}+\frac{m}{y}+\frac{n}{z}=0,$$

is satisfied by every set x_1, y_1, z_1; from the three equations so obtained l, m, n can be eliminated, and the result is $D=0$.

Hence the three derived elements I., II., III. are united with a single secondary element; they are related as stated by the theorems.

86. By means of Pascal's theorem any number of points on a conic 1 2 2′ 3 3′ can be linearly constructed.

For 23′, 2′3 determine I.; take any line through I. meeting 13′, 12′ in II., III.; then 3 II. and 2 III. intersect in 1′. If the tangent at 3 is to be constructed, 1′ must be made the same as

3, and then 31′ II. (*i.e.* 3 II.) is the tangent; the construction is therefore: 23′, 2′3 determine I.; (12′, 1′2 *i.e.*) 12′, 23 determine III.; hence the Pascal line I. III. is known, and this meets 13′ in II.; thus 3 II. is constructed.

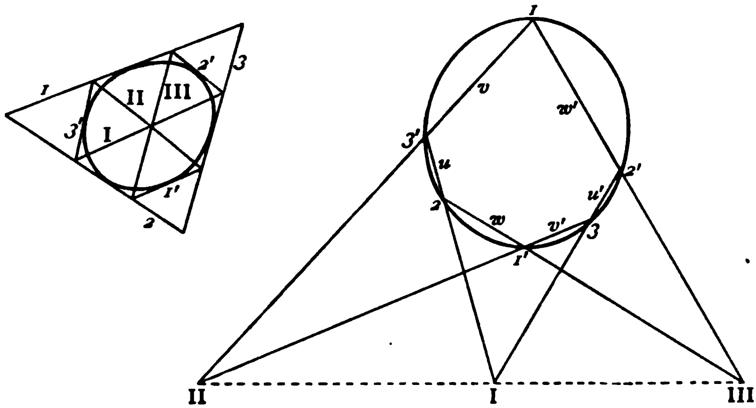

FIG. 26.

87. A better proof of these theorems depends on the equation of a conic with four given elements (§ 75); this proof (from Salmon's *Conic Sections*, § 267) is here given for Pascal's theorem; a few simple verbal changes will adapt it to Brianchon's theorem.

Let the six points taken in order (Fig. 26) be 1 3′ 2 1′ 3 2′, and let the sides be denoted by $u=0$, etc., as shown in the diagram.

Any conic through 1 3′ 2 1′ has for its equation

$$vw=kpu,$$

where $p=0$ is the line 1 1′. The conic F through the six points is one of this system; let the multipliers involved in u, v, w be adjusted so that this particular conic is

$$F=vw-pu=0.$$

Similarly any conic through 1 2′ 3 1′ is

$$v'w'=k'pu';$$

and the multipliers in u', v', w' being still undetermined may be chosen so that the particular conic F of this system has for its equation

$$F=v'w'-pu'=0.$$

Hence the expression F can be written in each of the forms

$$vw-pu, \quad v'w'-pu',$$

these are therefore identically the same; that is,

$$vw - pu = v'w' - pu',$$

from which $$vw - v'w' = p(u - u'),$$

showing that the expression $vw - v'w'$ splits up into factors, p and $u - u'$.

Hence the conic $vw - v'w' = 0$ is made up of the line $p = 0$ (*i.e.* 1 1′), and a line $u - u' = 0$, which is some line through I. But this conic passes through all the intersections of $vw = 0$ and $v'w' = 0$; that is, through the four points 1, 1′, II., III. The line $p = 0$ accounts for 1, 1′; the line $u - u' = 0$ must therefore account for II., III.; hence I., II., III. are collinear.

88. Chasles' Theorem is given in his *Traité des Sections Coniques* (1865), pp. 2, 3. He states it in a form involving symmetrically five tangents and five points:—

Four points on a conic determine at any fifth point of the conic a pencil whose cross-ratio is equal to that of the range determined on any fifth tangent by the tangents at the four points; *

and then deduces two fundamental properties of conics:— Four points on a conic determine at any fifth point a pencil of constant cross-ratio; and four tangents determine on any fifth tangent a range of constant cross-ratio: that is, *four elements determine with any fifth element a configuration of constant cross-ratio.*

The materials for a direct proof are contained in Examples 10-13 of Chapter III. The four points being 1, ±1, ±1 the conic is

$$ax^2 + by^2 + cz^2 = 0 \qquad (1),$$

with the condition $$a + b + c = 0 \qquad (2).$$

Hence one cross-ratio of the pencil determined by a fifth point x, y, z is $\dfrac{x^2 - y^2}{x^2 - z^2}$. Multiplying (2) by x^2, and subtracting from (1),

$$b(y^2 - x^2) + c(z^2 - x^2) = 0;$$

therefore $$\frac{x^2 - y^2}{x^2 - z^2} = -\frac{c}{b},$$

which is the same for all points x, y, z on the conic.

* "Si par quatre points d'une conique on mène les tangentes et quatre autres droites aboutissant à un cinquième point quelconque de la courbe : le rapport anharmonique de ces quatre droites sera égal à celui des quatre points de rencontre des quatre tangentes et d'une cinquième tangente quelconque."

This is perhaps as good an opportunity as any for acknowledging my general indebtedness to this fascinating work of M. Chasles.

The tangents at the four points are $ax \pm by \pm cz = 0$, *i.e.* $a, \pm b, \pm c$. Let any fifth tangent be $px + qy + rz = 0$, then

$$bcp^2 + caq^2 + abr^2 = 0 \ldots\ldots\ldots\ldots\ldots(3).$$

The cross-ratio determined on p, q, r by $a, \pm b, \pm c$ is

$$\frac{\dfrac{p^2}{a^2} - \dfrac{q^2}{b^2}}{\dfrac{p^2}{a^2} - \dfrac{r^2}{c^2}}.$$

Multiplying (3) by a, (2) by bcp^2, and subtracting,

$$c(a^2q^2 - b^2p^2) + b(a^2r^2 - c^2p^2) = 0;$$

therefore
$$\frac{\dfrac{p^2}{a^2} - \dfrac{q^2}{b^2}}{\dfrac{p^2}{a^2} - \dfrac{r^2}{c^2}} = -\frac{c}{b},$$

which is the same for all tangents p, q, r; and is the same as that found from the points.

A simple proof depending on a different special form of the equation of a conic is to be found in Salmon's *Conic Sections*, §§ 274, 275.

89. Chasles' two fundamental properties of conics are direct interpretations of the equation

$$\alpha\gamma = k\beta\delta,$$

$\alpha, \beta, \gamma, \delta$ being regarded (i.) as point coordinates, (ii.) as line coordinates.

(i.) For definiteness, let these be actual perpendiculars. Let the four points be A, B, C, D, and let any fifth point be P. Use (AB) to denote the angle subtended at P by A, B.

Then $\quad 2 \times \text{area } PAB = \alpha \times AB,$

$$\text{and also} = PA \,.\, PB \,.\, \sin(AB);$$

therefore
$$\begin{aligned} \alpha \,.\, AB &= PA \,.\, PB \,.\, \sin(AB), \\ \gamma \,.\, CD &= PC \,.\, PD \,.\, \sin(CD), \\ \beta \,.\, BC &= PB \,.\, PC \,.\, \sin(BC), \\ \delta \,.\, DA &= PD \,.\, PA \,.\, \sin(DA). \end{aligned}$$

Now
$$\frac{\alpha\gamma \,.\, AB \,.\, CD}{\beta\delta \,.\, BC \,.\, DA} = k\frac{AB \,.\, CD}{BC \,.\, DA},$$

and is therefore constant;

hence
$$\frac{\sin(AB)\sin(CD)}{\sin(BC)\sin(DA)} \text{ is constant};$$

that is, the pencil $\{P \,.\, ABCD\}$ is constant.

(ii.) Now considering tangents a, b, c, d, with a fifth tangent p, let (ab) denote the length intercepted on p by a, b. The equations of the intersections ab, bc, cd, da are $\xi=0$, $\eta=0$, $\zeta=0$, $\theta=0$, where ξ, η, ζ, θ are actual perpendiculars; the conic is $\xi\zeta=k\eta\theta$.

A diagram shows that

$$2 \text{ area } pab = \xi(ab),$$

$$\text{and also} = (ab)^2 \sin pa \sin pb \div \sin ab;$$

therefore

$$\begin{aligned} \xi \sin ab &= (ab) \sin pa \sin pb, \\ \zeta \sin cd &= (cd) \sin pc \sin pd, \\ \eta \sin bc &= (bc) \sin pb \cdot \sin pc, \\ \theta \sin da &= (da) \sin pd \sin pa. \end{aligned}$$

Now $\sin ab$, $\sin cd$, etc., are constant,

hence finally $$\frac{(ab)(cd)}{(bc)(da)} \text{ is constant,}$$

that is, the range $\{p \,.\, abcd\}$ is constant.

Note. The cross-ratio of the pencil determined in a conic by four points on the conic is spoken of as the cross-ratio of the points; but an implied reference to the conic must be understood, the cross-ratio being different for different conics through the four points.

EXAMPLES.

1. Taking two triangles in perspective, prove that the joins of non-corresponding vertices (six lines in all) touch a conic, and that the intersections of non-corresponding sides lie on a conic.

2. Show that two triangles that are conjugate with respect to a conic are in perspective.

3. Two triangles are self-conjugate with respect to a conic. Are they in perspective?

4. The six sides of two inscribed triangles touch a conic, and the six vertices of two circumscribed triangles lie on a conic.

5. Two triangles are self-conjugate with respect to a conic. Show that their six vertices lie on a conic, and that their six sides touch a conic.

6. Two triangles are to be drawn such that a conic can be described having each as a self-conjugate triangle. Choosing one triangle arbitrarily, how many of the determining elements of the other are at our disposal?

7. Considering the four points determined on a conic by two chords, prove that these points will be harmonic if the chords be conjugate.

8. Show that the problem "to draw through four points a conic such that the pencil determined in it by the four points shall be harmonic" has three real solutions if the four points be all real, two real solutions if they be all imaginary, one real solution if the points be two real and two imaginary.

9. Show that the three harmonic conics determined by four real points belong to the three different sets of the pencil.

Joachimsthal's Method.

90. The question of the common elements of two curves can be discussed with special facility when the coordinates of an element of the one curve are expressed in terms of a single parameter, while the other curve is represented by an equation. The general question as to the possibility of expressing the coordinates of a point on a curve parametrically is considered briefly in Chapter VIII.; but there is one case that can now be discussed with advantage, and used to exhibit the general method employed, viz., that in which the equation of one of the two curves is linear. For definiteness, point coordinates only will be explicitly referred to; the curves considered are therefore a curve of order n, and a straight line. The coordinates of a point on the line can always be expressed in terms of a single parameter; the points in which the line meets the curve are then determined by the values of the parameter given by an equation of degree n; and any particular relation of the line to the curve, being necessarily a particular relation of the points of intersection, is expressible by a relation in the roots of this equation.

91. Let the line be assigned by the two points on it, x', y', z'; x'', y'', z''; and let x, y, z divide the line joining x', y', z' to x'', y'', z'' in the ratio $m : l$ (or $1 : k$). Then $x : y : z = lx' + mx'' : ly' + my'' : lz' + mz''$.

Consider the intersections of this line and a curve $F = 0$.

I. Let $\quad F = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$

By substituting for x, y, z the expressions just given we limit ourselves to such points on $F = 0$ as are also on the line joining x', y', z' to x'', y'', z''; that is, to the intersections of the line and the conic. The resulting equation, arranged in terms of l, m, is

$$l^2(ax'^2 + by'^2 + \ldots) + 2lm(ax'x'' + \ldots) + m^2(ax''^2 + \ldots) = 0.$$

Writing x, y, z for x'', y'', z'', so that the points determining the line are x', y', z'; x, y, z, this equation becomes

$$l^2F' + 2lmp + m^2F = 0,$$

where p is written for

$$x(ax' + hy' + gz') + y(hx' + by' + fz') + z(gx' + fy' + cz').$$

The two values of $l : m$ given by this quadratic determine the two intersections of the line and the conic.

If x', y', z' be a point on the conic, $F' = 0$; the equation reduces to $2lmp + m^2F = 0$, one solution of which is $m : l = 0$. The second value of $m : l$ can be made to vanish by making the second intersection approach the first; as the equation must then reduce to $m^2F = 0$, the second intersection will approach the first indefinitely if $p = 0$. Hence if x, y, z satisfy the equation

$$x(ax' + hy' + gz') + y(hx' + by' + fz') + z(gx' + fy' + cz') = 0,$$

it lies on the tangent at x', y', z'; the equation of the tangent at x', y', z' is therefore the one just given.

II. Let F be of degree n. Write k for $l : m$, then the equation for intersections is

$$F(x + kx', y + ky', z + kz') = 0,$$

which, by Taylor's theorem, can be written

$$F(x, y, z) + k\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z}\right)F + \frac{k^2}{2!}(\quad)^2F + \ldots = 0.$$

Restoring $l : m$, and writing Δ for the operator

$$x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + z'\frac{\partial}{\partial z},$$

this becomes

$$m^nF + m^{n-1}l\Delta F + \frac{1}{2!}m^{n-2}l^2\Delta^2F + \ldots + \frac{1}{n!}l^n\Delta^nF = 0.$$

But by symmetry in x, y, z; x', y', z', this might have been obtained in the form

$$l^nF' + l^{n-1}m\Delta'F' + \frac{1}{2!}l^{n-2}m^2\Delta'^2F' + \ldots + \frac{1}{n!}m^n\Delta'^nF' = 0,$$

where Δ' is written for

$$x\frac{\partial}{\partial x'} + y\frac{\partial}{\partial y'} + z\frac{\partial}{\partial z'}.$$

It should be noticed that identically

$$\Delta^nF = n!F'\ ;\ \Delta^{n-1}F = \overline{n-1}!\Delta'F'\ ;\ \text{etc.}$$

If now $F' = 0$, one value of $m : l = 0$, and the point x', y', z' is on the curve. If in addition $\Delta'F' = 0$, a second value of $m : l = 0$; the line joining x', y', z' to x, y, z meets the curve in

two consecutive points at x', y', z', and is therefore a tangent. Hence the equation of the tangent at x', y', z' is

$$x\frac{\partial F}{\partial x'}+y\frac{\partial F}{\partial y'}+z\frac{\partial F}{\partial z'}=0. \quad \text{(See § 64.)}$$

92. If x', y', z' be not on the curve, consider the tangents drawn from x', y', z' to the curve. If the point of contact of any one of these be x_1, y_1, z_1, that tangent is

$$x\frac{\partial F}{\partial x_1}+y\frac{\partial F}{\partial y_1}+z\frac{\partial F}{\partial z_1}=0;$$

that is, $$\Delta_1 F_1=0.$$

This is to pass through the known point x', y', z'; hence x_1, y_1, z_1 must satisfy

$$x'\frac{\partial F}{\partial x_1}+y'\frac{\partial F}{\partial y_1}+z'\frac{\partial F}{\partial z_1}=0,$$

that is, x_1, y_1, z_1 must lie on the curve

$$\Delta F=x'\frac{\partial F}{\partial x}+y'\frac{\partial F}{\partial y}+z'\frac{\partial F}{\partial z}=0,$$

the first polar of x', y', z' with respect to F (§ 70). The first polar of x', y', z' with respect to this curve, viz., $\Delta(\Delta F)=0$, is called the second polar of x', y', z' with respect to F; it is plainly of order $n-2$.

Ex. Find, with respect to the curve $x^3-xz^2-y^2z=0$, the successive polars of (i.) $\frac{1}{3}$, 0, 1; (ii.) $\frac{2}{3}$, 0, 1; (iii.) 0, 0, 1.

93. In the case of the conic a special arrangement of the investigation is more interesting. There are, on the line considered, two pairs of points, viz., the determining points $Q(x, y, z)$, $Q'(x', y', z')$, and the points determined by the equation

$$l^2F'+2lmp+m^2F=0;$$

call these R, R'. Two pairs of points are susceptible of a particular relation of position; they may be harmonic. By the definition of harmonic division the two values of $l:m$ are numerically equal but opposite in sign; hence $\frac{l_1}{m_1}+\frac{l_2}{m_2}=0$; that is, the sum of the roots in the quadratic equation vanishes. Hence the condition that Q, Q' be harmonic conjugates with respect to the conic is

$$p=0,$$

that is,

$$x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz')=0,$$

and the locus of all points conjugate to Q' with respect to the conic is the straight line

$$p = x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz')=0.$$

This is called the polar of x', y', z'; and the line being here denoted by p, the point x', y', z' (Q') will now be called P.

All properties of poles and polars with respect to conics follow from this fundamental harmonic property:—

I. If the polar of P pass through Q, then P, Q are harmonic with respect to the conic, and therefore the polar of Q passes through P.

II. If R, R' be real, they separate and are separated by P, Q. Hence as R, R' approach one another indefinitely, the line $PRQR'$ becoming a tangent, one of the two points P, Q—Q, suppose—becomes coincident with RR'; hence the polar of P passes through the points of contact of tangents from P.

III. If P be on the curve, so coinciding with R, the condition of harmonic division requires that R' also coincide with R; and then Q is simply any point on the line PRR'; the locus of Q is therefore this line PRR', which is the tangent at P; that is to say, the polar of a point on the conic is the tangent at that point.

IV. The construction for the polar, given in § 78, is at once seen to be correct, owing to the harmonic properties of a complete quadrangle. Hence the theory of self-conjugate triangles (harmonic triangles) also follows.

94. If x', y', z' be not on the curve F, let it be required to make the line a tangent.

This requires that two roots of the equation in $l:m$ be made coincident. This applied to the special case of the conic gives the condition

$$FF' - p^2 = 0,$$

which is therefore the equation of the two tangents that can be drawn from x', y', z' to the conic $F=0$.

Ex. Determine the three tangents that can be drawn from 1, 1, 1 to the cubic $x^3+ky^2z=0$.

Note. It was shown in § 68 that this particular order-cubic is a class-cubic.

95. This method can also be applied to the determination of the intersections of two curves, by finding the lines joining any arbitrary point x', y', z' to the common points of the given curves $u=0$, $v=0$. Let these be of orders p, q; let a line through x', y', z' meet them in $P_1, P_2, \ldots, P_p$; $Q_1, Q_2, \ldots, Q_q$. For this line to pass through a common point A, one of the

points P must coincide with one of the points Q. But these two sets of points are determined by the two equations

$$l^p u' + l^{p-1} m\Delta' u' + \ldots = 0,$$
$$l^q v' + l^{q-1} m\Delta' v' + \ldots = 0;$$

hence the line considered will pass through an intersection A if these two equations have a common root. The condition is obtained by the elimination of $l:m$: it is therefore expressed in terms of $u', \Delta' u', \ldots; v', \Delta' v' \ldots$. Hence it involves x', y', z', a known point, and x, y, z, a variable point. It is therefore an equation in x, y, z, and by the mode of its formation it represents the lines joining x', y', z' to all the intersections of u, v.

Applying the process to two conics $u=0$, $v=0$, let the two polars of x', y', z' be p, q: the two equations are

$$l^2 u' + 2lmp + m^2 u = 0,$$
$$l^2 v' + 2lmq + m^2 v = 0,$$

and the equation obtained by eliminating $l:m$ is

$$(uv' - u'v)^2 = 4(uq - vp)(v'p - u'q);$$

this equation of degree 4 represents the four lines that join x', y', z' to the intersections of the conics u, v.

96. This method for dealing symmetrically with the intersections of straight lines and curves is due to Joachimsthal. It should be noticed that for its employment it may possibly not be essential that the coordinates of the various points be the same multiples of the actual perpendiculars, though they must of course belong to the same system.

For let the system of coordinates be given by

$$x, y, z = \lambda\alpha, \mu\beta, \nu\gamma.$$

Suppose that x', y', z' are equal to $\lambda\alpha', \mu\beta', \nu\gamma'$ multiplied by f', and that x'', y'', z'' are equal to $\lambda\alpha'', \mu\beta'', \nu\gamma''$, multiplied by f'', etc.

Then if α, β, γ divide α', β', γ'; $\alpha'', \beta'', \gamma''$; in the ratio $m:l$,

$$\alpha = \frac{l\alpha' + m\alpha''}{l+m}, \text{ etc.}$$

Hence, x, y, z being equal to $\lambda\alpha, \mu\beta, \nu\gamma$ multiplied by f,

$$\frac{x}{f\lambda} = \frac{l \cdot \dfrac{x'}{f'\lambda} + m\dfrac{x''}{f''\lambda}}{l+m}, \text{ etc.},$$

that is, $$x = \frac{f}{f'f''} \cdot \frac{lf''x' + mf'x''}{l+m}, \text{ etc.},$$

whence

$$x : y : z = lf''x' + mf'x'' : lf''y' + mf'y'' : lf''z' + mf'z'';$$

and writing l', m' for lf'', mf', these become

$$x:y:z=l'x'+m'x'':l'y'+m'y'':l'z'+m'z''.$$

Thus the effect of the different multipliers f is to alter all values of $l:m$ in the same ratio.

If then we propose to deal with these ratios directly, obtaining results that depend on their actual values (for example, if we wish a line to be bisected), these *different* multipliers f are not admissible; the coordinates of the various points must be the same multiples of the actual perpendiculars. But if the desired results depend only on a comparison of the values of $l:m$, the actual values themselves not entering into the final expressions, the different multipliers f have no effect; in this case it is not essential that the coordinates of the various points be the same multiples of the actual perpendiculars.

Examples.

1. Show that two lines can be drawn through any point so as to be harmonically divided by two conics: and that if the point be an intersection of the conics, the two lines are the tangents to the two conics at this point.

2. Hence show that the envelope of a line harmonically divided by two conics is a conic: and the common self-conjugate triangle being taken as triangle of reference, find the line equation of this envelope.

Find also the point equation. (Von Staudt. Salmon.)

3. Find the equation of this envelope if the equations of the conics be in the general form.

4. Show that any common tangent to the conics $u\pm\lambda v=0$ is cut harmonically by the conics $u=0$, $v=0$, for all values of λ; and apply this to find the equation of the envelope of a line harmonically divided by two conics. (F. Morley.)

5. Show that the locus of a point harmonically subtended*

* In the system of geometry in which the point is the primary element, the line the secondary element, the point is an entity, without parts, incapable of division; the line is an aggregate of points, and therefore capable of division; segments on a line are determined by their bounding points. If however the line be the primary element, it is the entity incapable of division; the point is regarded as an aggregate of lines, and therefore capable of division; the divisions being determined by the bounding lines. Hence linear magnitude and angular magnitude relate respectively to divisions of a line and of a point. Thus if two pairs of points on a line be harmonic, the line is harmonically divided; and if two pairs of lines through a point be harmonic, the point may be said to be harmonically divided. In Ex. 2 the line is spoken of as divided by the conics; hence without any explicit statement it is known that the conics

by two conics is a conic, passing through the eight points of contact of the common tangents to the two conics. Determine the point equation of this conic, the two given conics being referred to their common self-conjugate triangle. Find also the line equation. (Von Staudt. Salmon.)

Curves with Singular Points and Lines.

97. The process given for finding the tangent to any curve at a point (§ 64) involves the tacit assumption that only one line can be drawn to meet the curve in two indefinitely near points. And as a matter of fact, the process leads to an equation

$$x\frac{\partial F}{\partial x_1}+y\frac{\partial F}{\partial y_1}+z\frac{\partial F}{\partial z_1}=0,$$

which is determinate unless $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial y_1}, \frac{\partial F}{\partial z_1}$ vanish together. This same possibility presents itself as a source of disturbance in the determination of the reciprocal to a curve. The meaning of this will now be briefly considered.

98. Suppose that the assumption as to there being only one line through a point, meeting the curve in two consecutive points there, is not justified. Let there be two such lines, $u=0$, $v=0$. Then since u meets the curve in two points on v, the equation must be of the form

$$F=u\Phi+v^2\Psi=0.$$

Also $v=0$ is to give $u^2=0$, hence

$$u\Phi=u^2\mathrm{X}+uv\Theta,$$

and the equation of the curve F is

$$u^2\mathrm{X}+uv\Theta+v^2\Psi=0\text{......................}(1).$$

But this being the equation, every line through uv, viz., $u+\lambda v=0$, meets the curve in two points at uv. Hence if more than one line through a point meet the curve in two points there, then every line through the point has this property; the point is a double point. In the equation $u+\lambda v=0$, λ is still undetermined; it is therefore possible to make the line meet the curve in three points at uv; the

are regarded as point systems, *i.e.* as loci. In Ex. 5 the point might be spoken of as "harmonically divided by two conics"; the point being divided, the implied element is the line; the conics are regarded as line systems, and each has two line elements in common with the point. Thus there are two pairs of lines through the point, and these pairs are to be harmonic. But as these lines are tangents to the conic, the same idea may be conveyed by the phraseology adopted in the text.

equation for λ, exactly as in Cartesians, is a quadratic, and there are therefore two tangents at the double point. For example, if the equation of the curve be of the form

$$u^2U+uvV+v^3W=0,$$

$u=0$ gives $v^3=0$, and therefore $u=0$ is one tangent at the double point uv; and if the equation be of the form

$$u^3X+uvY+v^3Z=0,$$

$u=0$ gives $v^3=0$, and $v=0$ gives $u^3=0$; hence the lines $u=0$, $v=0$ are the two tangents at the double point uv.

The general principles of § 61 can be applied to the case of double points. The equation

$$F=u^2\mathrm{X}+uv\Theta+v^2\Psi=0$$

can be regarded as the final form of

$$uu'\mathrm{X}+uv\Theta+vv'\Psi=0,$$

when u, u' coincide, and also v, v'. Hence two of the points common to F, u coincide, or else are consecutive on v; and two of the points common to F, v coincide, or else are consecutive on u. These conditions are harmonized by the double point at uv.

In this argument it is not assumed that u, v are linear; they may be expressions of any degree, and what has just been proved is that every intersection of the curves $u=0$, $v=0$ is a double point on the curve

$$u^2\mathrm{X}+uv\Theta+v^2\Psi=0.$$

Ex. The curve

$$(y^2-x)^2+(y^2-x)(x-y)(y-1)+(x-y)^2(x^2+y^2)=0$$

has a double point at every intersection of $y^2-x=0$, $x-y=0$; that is, at 0, 0 and at 1, 1.

Similarly there is a triple point at uv if the equation be reducible to

$$u^3\Phi+u^2v\Psi+uv^2\mathrm{X}+v^3\Theta=0,$$

for every line $u+\lambda v=0$ now meets the curve in three points at uv; the tangents are defined as lines that meet the curve in four points, and are determined by a cubic equation in λ; there are therefore three tangents at the triple point.

99. Since the double point makes itself felt in the course of the work by rendering the process for finding the tangent nugatory, this must be by causing the equation

$$x\frac{\partial F}{\partial x_1}+y\frac{\partial F}{\partial y_1}+z\frac{\partial F}{\partial z_1}=0$$

to become illusory. The conditions for a double point at x, y, z are therefore

$$\frac{\partial F}{\partial x}=0, \quad \frac{\partial F}{\partial y}=0, \quad \frac{\partial F}{\partial z}=0.$$

Note. These are essentially the same as the ordinary Cartesian conditions. For let the Cartesian equation be $f(x, y)=0$; the conditions for a double point are

$$f=0, \quad \frac{\partial f}{\partial x}=0, \quad \frac{\partial f}{\partial y}=0.$$

Let the equation made homogeneous as directed in § 30 be $F(x, y, z)=0$. Then the condition $f=0$ gives $F=0$;

$$\frac{\partial f}{\partial x}=0, \text{ made homogeneous, gives } \frac{\partial F}{\partial x}=0;$$

$$\frac{\partial f}{\partial y}=0, \text{ made homogeneous, gives } \frac{\partial F}{\partial y}=0.$$

Also, by Euler's theorem of homogeneous functions,

$$x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}+z\frac{\partial F}{\partial z}=nF;$$

hence the condition $F=0$, with the help of $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, is reduced to

$$\frac{\partial F}{\partial z}=0.$$

The elimination of $x:y:z$ from these three homogeneous equations leaves one condition to be satisfied by the coefficients. Hence the general curve of any assigned order has not any double points (nodes or cusps); the presence of even one double point requires the equation to be specialized; a certain function of the coefficients must vanish.

Reciprocally, the general curve of any assigned class has not any double lines (double tangents or inflexional tangents); for the existence of these the equation must be specialized by the vanishing of a certain function of the coefficients.

Note. In this connection it may be remarked that if the reciprocal to the general curve of order m be of order n (the assigned general curve being of class n), it is not the *general* curve of order n. For example, the reciprocal to the general cubic is a sextic; now the general equation of a cubic involves 9 disposable constants; and the coefficients of the reciprocal sextic are expressed in terms of these 9 quantities. But in the general equation of a sextic there are 28 terms, and consequently 27 arbitrary constants. Thus the sextic is specialized; the 27 independent constants that belong to the general sextic are expressed in terms of 9 quantities, and by eliminating these 9 quantities in the various possible ways from the 27 equations, it is seen that certain functions of the coefficients of the sextic vanish.

The discussion of the tangents from a point (§§ 70, 92) shows that their points of contact are determined as inter-

sections of the curve and the first polar of the point, with the proviso that points at which $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$ vanish are not to be counted; the effect of these points is therefore to diminish the number of tangents that can be drawn from a point. These singular points have now been found to be double points; hence the presence of double points (nodes or cusps) on a curve of given order causes a diminution in the class; and the presence of double lines (double tangents or inflexional tangents) on a curve of given class causes a diminution in the order. For example, in § 68 two order-cubics were found to be, the one of class 6, the other of class 3; and it was shown in § 69 that one of these has a cusp. The investigation of the law of diminution belongs however to the theory of Higher Plane Curves.

100. The effect of an inflexion on the point equation, and reciprocally of a cuspidal tangent on the line equation, must be noticed. An inflexional tangent meets the curve in three points, instead of two; hence by the process of §§ 61, 98, if $u=0$ be an inflexional tangent whose point of contact is on $v=0$, the equation of the curve F must be

$$u\phi+v^3\psi=0.$$

101. As an example of the direct processes that can be employed consider a quartic.

This has a double point at A if the equation be of the form

$$y^2\phi+yz\psi+z^2\chi=0,$$

where ϕ, ψ, χ are general expressions of the second degree; hence there is a double point at A if the terms

$$x^4, \quad x^3y, \quad x^3z$$

be absent from the general equation of a quartic. Similarly there is a double point at B if the terms

$$y^4, \quad xy^3, \quad y^3z$$

be absent; and at C if the terms

$$z^4, \quad xz^3, \quad yz^3$$

be absent. Hence the most general equation of a quartic with double points at A, B, C is

$$ay^2z^2+bz^2x^2+cx^2y^2+2fx^2yz+2gxy^2z+2hxyz^2=0.$$

To determine the tangents at A, λ is chosen so that $y=\lambda z$ meets the curve in three points at A. The equation for intersections is

$$a\lambda^2z^4+2(g\lambda^2+h\lambda)xz^3+(c\lambda^2+2f\lambda+b)x^2z^2=0,$$

showing that for all values of λ, two intersections are given by $z=0$. To make z^3 a factor, λ must satisfy

$$c\lambda^2+2f\lambda+b=0.$$

Hence the tangents at A are determined by this equation; the equation of the tangents themselves is obtained by writing $\lambda=\frac{y}{z}$; hence the tangents at A, B, and C are

$$cy^2+2fyz+bz^2=0,$$
$$az^2+2gzx+cx^2=0,$$
$$bx^2+2hxy+ay^2=0.$$

The way in which the coefficients occur in these equations shows that these six lines are not independent; for the six ratios

$$c:f:b,\quad a:g:c,\quad b:h:a,$$

are connected by a relation

$$c:b\times a:c\times b:a=1.$$

There are however five independent ratios. Now five independent lines suggest a conic, and it will be found that the sixth line belongs to the same conic; this is most easily shown by determining a conic to touch all six lines.

Let the lines $cy^2+2fyz+bz^2=0$ be $my+nz=0$, etc. Then the coordinates of either line are 0, m, n; and since $\frac{y}{n}=\frac{z}{-m}$, m, n are determined by

$$cn^2-2fmn+bm^2=0.$$

Hence using line coordinates, the three pairs of elements are

$$\xi=0,\quad b\eta^2-2f\eta\zeta+c\zeta^2=0 \quad\ldots\ldots\ldots\ldots(1);$$
$$\eta=0,\quad c\zeta^2-2g\zeta\xi+a\xi^2=0 \quad\ldots\ldots\ldots\ldots(2):$$
$$\zeta=0,\quad a\xi^2-2h\xi\eta+b\eta^2=0 \quad\ldots\ldots\ldots\ldots(3);$$

and the question is, to determine a quadratic relation satisfied by these.

By (1), $\xi=0$ is to reduce the equation to

$$b\eta^2-2f\eta\zeta+c\zeta^2=0,$$

hence it must be $\xi\phi+b\eta^2-2f\eta\zeta+c\zeta^2=0$,

where the degree being 2, ϕ must be linear; therefore the equation is

$$\xi(A\xi+B\eta+C\zeta)+b\eta^2-2f\eta\zeta+c\zeta^2=0.$$

By the substitution of $\eta=0$, this is reduced to

$$A\xi^2+C\xi\zeta+c\zeta^2=0,$$

which by (2) must be the same thing as

$$a\xi^2-2g\xi\zeta+c\zeta^2=0.$$

Hence $A=a$, $C=-2g$. One quantity B is left, wherewith to

satisfy (3). But (3) simply requires that $A\xi^2+B\xi\eta+b\eta^2$ be the same as $a\xi^2-2h\xi\eta+b\eta^2$, where A is already known to be the same as a; hence taking $B=-2h$, all the conditions are satisfied. The conclusion is therefore:—

The quartic

$$ay^2z^2+bz^2x^2+cx^2y^2+2fx^2yz+2gxy^2z+2hxyz^2=0$$

has nodes at A, B, C; and the six nodal tangents touch the conic

$$a\xi^2+b\eta^2+c\zeta^2-2f\eta\zeta-2g\zeta\xi-2h\xi\eta=0.$$

EXAMPLES.

1. Writing the general equation of a cubic in the form

$$ax^3+by^3+cz^3+3a'x^2y+3b'y^2z+3c'z^2x$$
$$+3a''xy^2+3b''yz^2+3c''zx^2+6dxyz=0,$$

show that the conditions for a double point at C are $c=0$, $c'=0$, $b''=0$; and that CA, CB will be the tangents at this double point if $b'=0$, $c''=0$.

2. Show that the conditions for a cusp at C are $c=0$, $c'=0$, $b''=0$, $d^2=b'c''$; and that if $x-y=0$ be the tangent at the cusp, $c''=b'$, $d=-b'$.

3. Find the equation of a cubic, touching the lines $u=0$, $v=0$, $w=0$ on the line $s=0$. Hence show that, P, Q, R being collinear points on a cubic, the points P', Q', R' in which the tangents at P, Q, R meet the cubic again are collinear.

4. Find the equation of a cubic having $y=0$, $z=0$, as inflexional tangents at points lying on the line $x+y+z=0$. Show that this cubic has a third inflexion, also on the line

$$x+y+z=0.$$

5. Find the equation of a cubic with a cusp at vw, tangent to $w=0$, and an inflexion at uv, tangent to $u=0$.

6. Show that a quartic can be found having cusps at A, B, C; and that the cuspidal tangents are concurrent.

CHAPTER VI.

METRIC PROPERTIES OF CURVES; THE LINE INFINITY.

Introductory.

102. The fundamental identical relation in point coordinates leads us to consider a certain straight line lying entirely at infinity. In considering therefore properties of curves that depend on the actual values of the coordinates, that is, on the identical relation in point coordinates, we naturally consider the relation of the curve to this special line. This line being specialized only in position, not in nature, all general investigations on the relation of a curve to a line are applicable.

Points at Infinity. Asymptotes.

103. In the first place let the curve considered be a conic. The only specialization in the relation of a conic to a line is that expressed by contact; the two points of intersection coincide. Hence a classification of conics presents itself, according as the line infinity is or is not a tangent. One division of conics into species has already been found, viz., into proper and degenerate; that however is simply a cross-division; two straight lines, not parallel, give distinct points at infinity; parallel straight lines give coincident points. But as regards the points at infinity, the eye introduces another distinction, that between real and imaginary; this division is in a sense accidental, but is recognized in the classification of conics.

These fall therefore into three species:—

I. (i.) *The points at infinity being coincident, the conic is a parabola.*

II. (ii.) *The points at infinity being real and distinct, the conic is a hyperbola.*

II. (iii.) *The points at infinity being imaginary, the conic is an ellipse.*

Hence a line-pair comes under the heading parabola, hyperbola, ellipse, according as the lines are parallel, real and distinct, or imaginary.

Ex. Using trilinears, the conics represented by

$$fyz+gzx+hxy=0,$$

are distinguished by means of

$$ax+by+cz=0.$$

The intersections are given by

$$bhy^2+(bg+ch-af)yz+cgz^2=0$$

hence the conic is a hyperbola, a parabola, or an ellipse, according as

$$a^2f^2+b^2g^2+c^2h^2-2bcgh-2cahf-2abfg$$

is $+$, 0, or $-$.

Ex. Using areals, the nature of the conic depends on

$$f^2+g^2+h^2-2gh-2hf-2fg.$$

104. Hence to be told that a conic is a parabola is to be given one simple condition; for one tangent is given. Consequently (§ 83) four more tangents determine the conic uniquely, four points determine it as one of two; thus one parabola can be drawn to touch four lines; two parabolas can be drawn to pass through four points.

Ex. 1. The line infinity being $a_0x+b_0y+c_0z=0$, find the condition that the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

be a parabola. Deduce the ordinary condition in Cartesians.

Ex. 2. Using actual perpendiculars for line coordinates, find the condition that the general envelope of the second class be a parabola.

105. Any curve of order n cuts the line infinity in n points; some may be real, some imaginary; and there may be coincidences. The position of any point on the line infinity is indicated by its direction, all lines in this direction passing through the point. If the curve *touch* the line infinity at P, no one of these parallel lines (lines through P) has any special relation to the curve; but if the curve *cut* the line infinity at P, the tangent is a line through P, distinct from the line infinity, and is therefore one of the system of parallel lines, that is, it is an ordinary line. Such a line is called an asymptote to the curve; an asymptote is therefore defined as *a tangent whose point of contact is at infinity, the tangent itself not lying entirely at infinity.* Thus the definition excludes the line infinity from the asymptotes. A parabola has no asymptotes, there are no points where the curve simply *cuts* the line infinity; a hyperbola and an ellipse have each two asymptotes, respectively real and imaginary. A curve of order n may have

n asymptotes; but this number is diminished by contacts with the line infinity. If k points be used in accounting for these contacts, the number of asymptotes is $n-k$.

106. Let $s=0$ be the line infinity, and let x', y', z' be its pole with regard to a conic $F=0$; by § 74 the equation of the asymptotes of the conic is

$$FF'-s^2=0.$$

If now s be a tangent, the pole is on s, and also on F, hence $F'=0$, and the equation reduces to $s^2=0$, the line infinity counted twice, but not to be counted as an asymptote.

Ex. Using areals, find the asymptotes of

$$2yz+2zx+2xy=0 \quad\text{.......................(1).}$$

The line infinity is $$x+y+z=0 \quad\text{.......................(2).}$$

The polar of a point x', y', z' is

$$x(y'+z')+y(z'+x')+z(x'+y')=0,$$

and this is the same as (2) if

$$y'+z'=z'+x'=x'+y'=1,$$

whence $$x'=y'=z'=\tfrac{1}{2}.$$

Hence the asymptotes are

$$(2yz+2zx+2xy)(2.\tfrac{1}{2}.\tfrac{1}{2}+2.\tfrac{1}{2}.\tfrac{1}{2}+2.\tfrac{1}{2}.\tfrac{1}{2})-(x+y+z)^2=0,$$

that is, $$(x+y+z)^2-3(yz+zx+xy)=0,$$

that is, $$x^2+y^2+z^2-yz-zx-xy=0.$$

But in any particular example it is generally more convenient to adopt the other process of § 74, and simply express that

$$F+ks^2=0$$

is a pair of straight lines.

Ex. $$k(x+y+z)^2+2(yz+zx+xy)=0$$

is a line-pair if $$abc+2fgh-af^2-bg^2-ch^2=0,$$

that is, if $$k^3+2(k+1)^3-3k(k+1)^2=0,$$

which gives simply $$3k+2=0.$$

Hence the asymptotes are, as before,

$$x^2+y^2+z^2-yz-zx-xy=0,$$

which may be written

$$(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)=0,$$

where ω is an imaginary cube root of unity.

If then the conic be not a parabola, it has a pair of asymptotes, real or imaginary, u, u'; and the equation of these is

$$uu'=F+ks^2=0$$

when k is properly determined.

Hence $$F=uu'-ks^2.$$

When this is expressed in Cartesians, the term ks^2 is a constant; hence, as is known, *the equation of a conic, $F=0$, differs from the equation of its asymptotes, $uu'=0$, only by a constant.*

107. Similarly for curves of higher order. Let $F=0$ be a cubic, having its three points at infinity distinct; there are therefore three asymptotes, $t_1=0$, $t_2=0$, $t_3=0$. By the principle of § 61 a cubic touching these three lines on the line $s=0$ has an equation of the form

$$F=t_1t_2t_3+s^2l=0,$$

where l is a linear function. In passing to Cartesians, s becomes a constant, and the equation is

$$F=t_1t_2t_3+\text{a linear function}=0.$$

Hence the equation of the cubic $F=0$ differs from the equation of its asymptotes $t_1t_2t_3=0$ only in the terms of degree $\ngtr 1$. Moreover, t_1, t_2, t_3 meet the cubic again where $l=0$; that is, the (finite) points in which the asymptotes of a cubic cut the curve lie on a straight line $l=0$. (Compare Ex. 3 at the end of Ch. V.)

A quartic with all four points at infinity distinct has for its equation

$$F=t_1t_2t_3t_4+s^2u_2=0,$$

showing that the intersections of the curve and its asymptotes lie on a conic $u_2=0$; and that the Cartesian equation of the quartic differs from that of its asymptotes only in the terms of degree $\ngtr 2$. And in general $F_n=t_1t_2t_3\ldots t_n+s^2u_{n-2}=0$ is the equation of a curve of order n with n distinct points at infinity; hence the remaining intersections of the curve and its asymptotes lie on a curve of order $n-2$, $u_{n-2}=0$; and the Cartesian equations of the curve and its asymptotes are alike as to the terms of the two highest degrees.

108. From the general principles here used (§§ 61, 98, 107) the ordinary rules for the determination of asymptotes in Cartesian coordinates follow at once. Let the triangle of reference be that made by the Cartesian axes, $x=0$, $y=0$, and the line infinity, $z=0$. Let the equation be arranged according to powers of z; denote the terms of degree n in x, y by u_n, etc. Then the equation of the curve $F=0$ is

$$u_n+zu_{n-1}+z^2u_{n-2}+\ldots+z^nu_0=0.$$

Since u_n is a homogeneous expression in x, y, it splits up into n factors, $l_1, l_2, \ldots, l_n$; now the lines joining xy to the intersections of $F=0$ and $z=0$ are given by $u_n=0$; that is, by $l_1=0$, $l_2=0, \ldots, l_n=0$.

I. Let l_1 be a non-repeated factor of u_n, and let the point l_1z be L_1. There is therefore through L_1 an asymptote, whose equation is

$$l_1+\mu z=0,$$

where μ is to be determined so that the line shall be a tangent with its point of contact on $z=0$. The equation $F=0$ must therefore be of the form

$$(l_1+\mu z)V+z^2W=0.$$

(*a*) If now l_1 be a factor in u_{n-1}, the equation is at once expressible in this form; for u_n and u_{n-1} contain l_1, and the remaining terms contain z^2; hence the equation of the curve is

$$l_1V+z^2W=0,$$

and consequently the line $l_1=0$ is itself the asymptote. Hence the rule:—A non-repeated factor of u_n, if also a factor of u_{n-1}, gives an asymptote when equated to zero.

(*b*) If however l_1 be not a factor in u_{n-1}, μ must be determined by substitution, as in the ordinary Cartesian process.

II. But if any factor in u_n be repeated, l suppose, so that u_n is l^2v_{n-2}, the equation is

$$F=l^2v_{n-2}+zu_{n-1}+z^2u_{n-2}+\ldots+z^nu_0=0.$$

(*a*) If l be not a factor in u_{n-1}, this equation is of the form

$$l^2V+zW=0,$$

showing that $z=0$ is the tangent at the point lz, that is, the line infinity is itself the tangent. Hence a repeated factor of u_n, which is not a factor of u_{n-1}, indicates contact with the line infinity in the direction determined by equating the factor to zero.

(*b*) If l be a factor in u_{n-1}, the equation is of the form

$$l^2V+zlU+z^2W=0,$$

showing that there is a double point at lz. There are therefore two lines through lz to be determined, $l+\mu_1z=0$, $l+\mu_2z=0$; and these are to be determined so as to have three points in common with the curve.

Thus the rules for determining asymptotes in Cartesians are:—

(i.) *Any non-repeated factor l of u_n indicates an asymptote $l+\mu=0$; this will pass through the origin C,* (i.e. $\mu=0$) *if l be a factor in u_{n-1}.*

(ii.) *Any repeated factor l of u_n, if not a factor of u_{n-1}, indicates contact with infinity in the direction $l=0$.*

(iii.) *Any repeated factor l of u_n, if a factor of u_{n-1}*

indicates a double point at infinity in the direction $l=0$; and there are two parallel asymptotes to be determined.

Similarly a three-fold factor in u_n is accounted for in different ways according as it occurs or not in u_{n-1}, u_{n-2}; and in general the explanation is more easily seen when the equation is thrown into the homogeneous form.

Diameters and Centre of a Conic.

109. From a line and a conic a particular point is derived, viz., the pole of the line with respect to the conic. We have therefore to consider with respect to a conic a special point, the pole of the line infinity; and in connection with this, the polars of all points at infinity, these polars being concurrent in the special point.

Let P be any point at infinity, and consider chords KK' passing through P. If any such chord meet the polar of P in P', we know that KK' is harmonically divided in PP'. Hence P being at infinity, KK' is bisected at P'; that is, *the locus of the bisections of parallel chords is a straight line,* the polar of the point at infinity through which the chords pass. Any such line is called a diameter; and as diameters are the polars of points at infinity, that is of collinear points, *all diameters are concurrent in R, the pole of the line infinity.* Here two cases must be distinguished according as the line infinity (1) is, (2) is not, a tangent to the conic.

(1) R is on the line infinity; all diameters of a parabola are parallel.

(2) R is at a finite distance; all diameters of an ellipse or hyperbola meet in a point R. The defining property of diameters shows that every chord through R is bisected there; the point R is called the centre of the conic. Thus the classification of conics based on their position with regard to the line infinity is exactly indicated by the terms central (ellipse and hyperbola) and non-central (parabola).

To determine the centre of a conic it is necessary simply to find the pole of the line infinity. If this line be

$$a_0x+b_0y+c_0z=0,$$

comparison with the equation of the polar of x', y', z' with respect to the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

viz., with

$$x(ax'+hy'+gz')+y(hx'+by'+fz')+z(gx'+fy'+cz')=0,$$

shows that the coordinates of the centre are given by the equations

$$\frac{ax+hy+gz}{a_0}=\frac{hx+by+fz}{b_0}=\frac{gx+fy+cz}{c_0};$$

hence the centre is uniquely determined.

For the transition to Cartesians, the line infinity is $z=0$, that is,

$$0\,.\,x+0\,.\,y+z=0;$$

hence $a_0=0$, $b_0=0$; and the equations found become

$$ax+hy+g=0,$$
$$hx+by+f=0,$$

the ordinary Cartesian equations for the centre of a conic.

110. The conjugate relation of points (or lines) leads to the theory of conjugate diameters. In the general theorem (§ 73) "if the polar of P pass through Q, the polar of Q passes through P," let P, Q be at infinity; their polars p, q become diameters intersecting in R; since p passes through Q, it is the line RQ, and q is the line RP; hence RP bisects chords parallel to RQ, and RQ bisects chords parallel to RP.

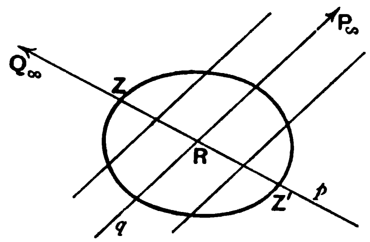

FIG. 27.

Thus the diameters are arranged in pairs of *conjugate diameters* (Fig. 27). Let p (*i.e.* RQ) meet the conic in Z, Z', then the tangents at Z, Z' pass through P, that is, they are parallel to the chords bisected. Thus all the ordinary properties of conjugate diameters follow from the theory of poles and polars.

111. RPQ is a self-conjugate triangle; hence taking it as triangle of reference, the conic is

$$px^2+qy^2+rz^2=0,$$

where $z=0$ is the line PQ, that is, the line infinity. The transition to Cartesians is therefore accomplished by writing $z=1$; the equation becomes

$$px^2+qy^2+r=0;$$

hence the equation of a central conic referred to any pair of conjugate diameters as Cartesian axes is

$$px^2+qy^2=\text{constant}.$$

Now the two central conics, ellipse and hyperbola, are discriminated by means of the points at infinity; these are given by

$$z=0, \quad px^2+qy^2+rz^2=0;$$

that is, by $$px^2+qy^2=0.$$

They are therefore imaginary, that is, the conic is an ellipse, if p and q have the same sign; and the conic is a hyperbola if p and q have opposite signs. Hence by taking *any* pair of conjugate diameters of an ellipse as axes the equation is reduced to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

a form which shows that all diameters meet the ellipse in real points; and by taking any pair of conjugate diameters of a hyperbola as axes, the equation becomes

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

showing that conjugate diameters meet the curve, one in real points, one in imaginary points.

This reduction of the equation, depending on the point R, is inapplicable to the parabola. But in this case taking for $x=0$ any tangent, and for $y=0$ the line joining the point of contact of this tangent to the point of contact of the line infinity (*i.e.* of $z=0$), the equation becomes

$$y^2=kxz;$$

hence the Cartesian equation of the parabola, with any diameter and the tangent at its vertex as axes, is

$$y^2=px.$$

EXAMPLES.

1. An imaginary conic can have real points in number 4, 2, or 0.

2. Write down the equation of

(i.) an imaginary ellipse;
(ii.) an imaginary parabola;
(iii.) an imaginary hyperbola.

3. Determine the locus of the centre of a conic through four points (*i.e.* the centre-locus of a pencil of conics).

4. Determine the centre-locus of a range of conics.

5. Show that to be given the centre of a conic amounts to two conditions.

6. From the general condition for conjugate lines deduce the ordinary Cartesian condition for conjugate diameters, the centre being origin.

7. Show that in the case of a pencil of conics through four real points, two of the three sets must necessarily be composed of hyperbolas.

8. Show how (*a*) the three line-pairs,
(*b*) the two parabolas,

present themselves in the pencil whose fundamental points are

(1) imaginary,

(2) real; noticing especially the case of the points forming a parallelogram.

CHAPTER VII.

METRIC PROPERTIES OF CURVES; THE CIRCULAR POINTS.

Two Special Imaginary Points at Infinity.

112. Before entering on the discussion of properties of curves that depend on the fundamental identical relation in line coordinates, the investigation of the significance of this relation referred to in § 29 must be given.

In dealing with point coordinates it was found that although in general x, y, z are subject to a condition of inequality which in trilinears is

$$ax+by+cz \neq 0,$$

yet there are points not conditioned by this, viz., all points lying on a certain straight line. Similarly ξ, η, ζ are subject to a condition of inequality which may be written

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C \neq 0;$$

but certainly one line exists not conditioned by this, viz., $ax+by+cz=0$, for which $\xi, \eta, \zeta=a, b, c$.

Note. Using areals this line is $x+y+z=0$, whence $\xi, \eta, \zeta=1, 1, 1$, values which similarly do not satisfy the corresponding condition of inequality,

$$a^2\xi^2+b^2\eta^2+c^2\zeta^2-2bc\cos A.\eta\zeta-2ca\cos B.\zeta\xi-2ab\cos C.\xi\eta \neq 0.$$

Since then at any rate *one* line exists not subject to the ordinary condition, the course that naturally suggests itself is to consider *all* the lines that escape this condition; that is, the lines whose coordinates make

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C=0.$$

This equation being of the second degree, all the lines now considered form a particular system of the second class. But the expression on the left splits up into linear factors; the system therefore degenerates into two of the first class, and the envelope is a pair of points. Consequently

the exceptional lines under consideration pass through one or other of two fixed points, $\omega=0$, $\omega'=0$, where ω, ω' are the linear factors of the expression on the left. These factors being imaginary, the points are conjugate imaginary points; the line joining them is real, and its coordinates, being given by

$$\xi-\cos C\,.\,\eta-\cos B\,.\,\zeta=0,$$
$$-\cos C\,.\,\xi+\eta-\cos A\,.\,\zeta=0,$$

(see § 71, and Ex. 4, after § 74) are

$$\xi:\eta:\zeta=\sin A:\sin B:\sin C=a:b:c;$$

the line is therefore $ax+by+cz=0$, the special line at infinity; the two imaginary points ω, ω' are at infinity. Since only one real line can pass through an imaginary point, all lines through ω, ω', other than the line infinity, are imaginary; through any real point ρ two of these lines pass, viz., $\rho\omega$, and $\rho\omega'$; and these are conjugate imaginary lines.

Thus corresponding to the statement of § 27 relating to point coordinates, the conclusion here arrived at may be stated as follows:—

In general the coordinates of a line are subject to a quadratic inequality,

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C \neq 0,$$

but there are lines at variance with this condition; these are the totality of lines passing through one or other of two fixed imaginary points at infinity; through every point in the plane there pass two of these exceptional lines.

Note. A remark analogous to that in the Note to § 27 may here be made; but the result is better stated in a more general form with reference to any quadratic expression in line coordinates, not necessarily a product of linear factors. If this expression be ϕ, taking any line p at random, this does not touch the conic ϕ; that is, in general the coordinates of a line in the plane make $\phi \neq 0$; but there are lines for which $\phi=0$, viz., the totality of lines touching the conic ϕ. The importance of this extension will appear in Chapter XII.

113. The coordinates of the two exceptional lines through any point ρ are obtained by combining the equations

$$\rho=0, \text{ that is, } \lambda\xi+\mu\eta+\nu\zeta=0,$$

and $$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C=0.$$

Thus the two lines through C are obtained from the last equation combined with $\zeta=0$; that is, from

$$\zeta=0, \text{ and } \xi^2-2\xi\eta\cos C+\eta^2=0 \quad\text{.............(1).}$$

Solving these, we have the coordinates of the lines; if however their equations be required, either one is $\xi x+\eta y=0$,

where ξ, η, are determined from (1); the equation of the two is therefore found by writing

$$\xi : \eta = y : -x \text{ in (1);}$$

hence it is

$$x^2 + 2xy \cos C + y^2 = 0 \text{ ....(2).}$$

As a special case, let C be a right angle; equation (2) becomes $x^2 + y^2 = 0$. Hence in rectangular Cartesians, the lines joining the origin to ω, ω' are $x^2 + y^2 = 0$.

If we attempt to treat these exceptional lines as ordinary lines, and regard them as having direction, we are led to a paradoxical result. The angle made by a line $y = kx$ with $y = mx$ being θ, the ordinary formula gives $\tan\theta = \dfrac{k-m}{1+km}$. If now $y = kx$ be one of the lines $x^2 + y^2 = 0$, so that $k = i$, this becomes

$$\tan\theta = \frac{i-m}{1+im} = \frac{i(i-m)}{i-m} = i.$$

Thus m does not appear in the expression for $\tan\theta$.

If we interpret this result without considering the meaning of the symbols, it is $\tan\theta = \text{constant}$; that is, the exceptional element makes the same angle with *all* lines; with reference to this interpretation of the result $\tan\theta = i$, the exceptional lines are called *isotropic*. Thus an isotropic line is any line through either of the points ω, ω'.

But considering the meaning of the symbols, the question arises, Can we regard the angle as in any way defined by the equation $\tan\theta = i$? We shall not enter on the question here suggested, and shall avoid the necessity of this by refraining from associating the idea of direction with these lines, though the name isotropic will be used.

Note. The meaning of this indeterminateness appears in § 275.

114. By means of the expression

$$\frac{2\Delta(x\xi + y\eta + z\zeta)}{(ax+by+cz)(\xi^2 + \eta^2 + \zeta^2 - 2\eta\zeta \cos A - 2\zeta\xi \cos B - 2\xi\eta \cos C)^{\frac{1}{2}}}$$

for the distance from the point x, y, z to the line ξ, η, ζ, it was proved in § 28 that the distance from any point to the special line infinity is a constant, infinite in value.

Similarly the product of the distances to any line from the special points ω, ω' is a constant, infinite in value. The point x, y, z being regarded as fixed, and the line ξ, η, ζ as variable, the factor

$$\frac{2\Delta}{ax+by+cz}$$

in the above expression is a constant, k. Hence the distance from ω to ξ, η, ζ is

$$k\frac{\omega}{(\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C)^{\frac{1}{2}}};$$

and the distance from ω' to ξ, η, ζ is

$$k'\frac{\omega'}{(\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C)^{\frac{1}{2}}}.$$

The product of these distances is therefore

$$kk'\frac{\omega\omega'}{\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C}.$$

Now the linear expressions ω, ω' are by definition the factors of

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C;$$

hence the numerator and denominator of the fraction involved in the expression just written are identically the same. The product of the distances is therefore kk', which does not involve the coordinates of the variable line, and is therefore an absolute constant. It is infinite, for x, y, z referring to ω, make $ax+by+cz=0$; hence k is infinite, as also k'.

Condition of Perpendicularity.

115. The fact that there are two exceptional lines through any point shows that a pair of ordinary lines may have a special relation. For the ordinary lines and the exceptional lines form two pairs; and these may be harmonic. Thus, for example, using rectangular Cartesians, the lines $ax^2+2hxy+by^2=0$ are harmonic with regard to the isotropic lines through their intersection, $x^2+y^2=0$, if (§ 47) $a+b=0$; but this is the Cartesian condition of perpendicularity. We have used the conception of perpendicularity at intervals in the preceding pages, chiefly in illustrative examples, but have not hitherto formulated this conception, nor given any systematic discussion of properties dependent on it. It now however presents itself naturally as expressing the special relation that a pair of lines may hold with regard to the exceptional elements, and we are lead to the definition:—

Lines harmonic with respect to the isotropic lines through their intersection are said to be perpendicular.

Considering the pencil formed by these two pairs of lines as cut by the line infinity, this may be otherwise stated:—Lines that divide $\omega\omega'$ harmonically are perpen-

dicular. Or again, remembering that ω, ω' present themselves as a degenerate conic, and recalling the harmonic properties of conjugate points and lines (§ 78):—Lines conjugate with regard to

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C=0$$

are said to be perpendicular.

Hence the lines
$$l_1x+m_1y+n_1z=0,$$
$$l_2x+m_2y+n_2z=0,$$
where the point coordinates are trilinears, that is, the lines l_1, m_1, n_1; l_2, m_2, n_2; are perpendicular if

$$l_1l_2+m_1m_2+n_1n_2-(m_1n_2+m_2n_1)\cos A$$
$$-(n_1l_2+n_2l_1)\cos B-(l_1m_2+l_2m_1)\cos C=0.$$

Ex. Find the condition that two lines whose equations are given in areals be perpendicular.

Relation of a Conic to the Special Points.

116. Just as conics were differentiated by their relation to the line infinity, they may be further differentiated by their relation to ω, ω'. These being conjugate imaginary points, a real conic through one passes through the other also. The principal distinction is therefore that the conic

(1) passes through ω, ω';
(2) does not pass through ω, ω';

but since there are two pairs of points at infinity to be considered (viz., ω, ω', and the intersections of the line infinity and the conic) taking into account the special relation that may be held by these two pairs, we have to consider that the conic may divide $\omega\omega'$ harmonically.

The Circle.

117. Since the points ω, ω' are imaginary, a conic through ω, ω' is an ellipse, by the broad classification of § 103. It is a *special* ellipse, and we are now considering sub-divisions. Any conic through ω, ω' has a line equation of the form

$$\rho^2=k\omega\omega',$$

where $\rho=0$ is the pole of the line $\omega\omega'$ (that is, the line infinity), and is therefore the centre of the conic. Substituting for ρ, $\omega\omega'$ their expressions in terms of ξ, η, ζ this becomes

$$(f\xi+g\eta+h\zeta)^2=k(\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C).$$

Here f, g, h are the point coordinates of the centre ρ; the

equation can be made homogeneous in f, g, h by means of the identical relation

$$af+bg+ch=2\Delta;$$

it is then

$$\frac{4\Delta^2(f\xi+g\eta+h\zeta)^2}{(af+bg+ch)^2(\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C)}=k.$$

Writing r^2 for k, this expresses that the distance from the point f, g, h to the line ξ, η, ζ is constant and equal to r. Hence the conic through ω, ω' is characterized by the property that all its tangents are at the same distance r from the centre. It is therefore a circle; and the conclusion is:—*Every conic through ω, ω' is a circle;* and the work being reversible:—*Every circle is a conic through ω, ω'.* The line equation of a circle with centre $\rho=0$, radius r, is therefore

$$\rho^2=r^2\omega\omega'.$$

Two special cases suggest themselves:—

(i.) Let r be indefinitely small; the equation reduces to $\rho^2=0$; that is, the indefinitely small circle regarded as an envelope of the second class is simply the point ρ counted twice; it has therefore no particular bearing on the present investigation.

(ii.) Let r be indefinitely great; the equation reduces to $\omega\omega'=0$; that is, the infinitely great. circle regarded as an envelope breaks up into a pair of imaginary points at infinity. Hence the degenerate envelope

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C=0$$

may be regarded as a circle of infinite radius.

118. The points ω, ω' which have now been shown to lie on every circle are called the *circular points*, and the isotropic lines are often called circular lines. The circular points are frequently referred to as I, J.

To be told that a conic is a circle amounts therefore to two conditions; for two points on the conic are given. Hence, as in § 83, three points determine a circle uniquely; three tangents determine the circle as one of four.

The form of the line equation

$$\rho^2=k\omega\omega'$$

shows that $\rho\omega, \rho\omega'$ are tangents at ω, ω'. Hence concentric circles, $\rho^2=k\omega\omega', \quad \rho^2=k'\omega\omega',$ have double contact at infinity (viz. at ω, ω').

119. The trilinear coordinates of ω, ω' are at once found from their equations. Writing

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C$$

in the form $(-\xi+\eta\cos C+\zeta\cos B)^2+(\eta\sin C-\zeta\sin B)^2$, these equations are seen to be

$$-\xi+\eta(\cos C+i\sin C)+\zeta(\cos B-i\sin B)=0,$$
$$-\xi+\eta(\cos C-i\sin C)+\zeta(\cos B+i\sin B)=0;$$

that is,
$$-\xi+\eta e^{iC}+\zeta e^{-iB}=0,$$
$$-\xi+\eta e^{-iC}+\zeta e^{iB}=0;$$

hence the coordinates of the circular points are

$$(-1,\ e^{iC},\ e^{-iB}),\ (-1,\ e^{-iC},\ e^{iB}),$$

which may also be written in either of the forms

$$(e^{-iC},\ -1,\ e^{iA}),\qquad (e^{iC},\ -1,\ e^{-iA});$$
$$(e^{iB},\ e^{-iA},\ -1),\qquad (e^{-iB},\ e^{iA},\ -1).$$

120. From the fact that all circles pass through the same two points at infinity, it follows that the equation of a circle in point coordinates is of a special form. Let $s=0$ be the line infinity, and let $S=0$ be any one circle; any conic through the intersections of S and s has an equation of the form

$$S'=S+sv=0,$$

where v is linear; hence the equation of any circle is

$$S+sv=0,$$

where $S=0$ is any particular circle. Thus all that is necessary is to find the equation of some one circle; for example, the circumscribing circle.

Any conic through A, B, C is

$$fyz+gzx+hxy=0;$$

this is a circle if it pass through the points ω, ω', whose coordinates were found in the last section. Writing the equation in the form

$$\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0,$$

these conditions give

$$-f+ge^{-iC}+he^{iB}=0,$$
$$-f+ge^{iC}+he^{-iB}=0;$$

therefore
$$g(e^{iC}-e^{-iC})=h(e^{iB}-e^{-iB}),$$

that is,
$$g\sin C=h\sin B;$$

whence $\quad f:g:h=\sin A:\sin B:\sin C=a:b:c,$

and the equation in trilinears of the circumscribing circle is

$$ayz+bzx+cxy=0;$$

giving, for the equation of any circle,

$$ayz+bzx+cxy+(ax+by+cz)(lx+my+nz)=0.$$

121. All these results can be obtained by point coordinates. The equation of the circumscribing circle can be found by means of the property that the angle in a semicircle is a right angle. Drawing BA', CA' perpendicular to BA, CA, their intersection A' is on the circle; the coordinates of A' are connected by the relations

$$-\alpha=\gamma\cos B,\quad -\alpha=\beta\cos C \text{ (from Fig. 28)},$$

and are therefore $$-1,\ \frac{1}{\cos C},\ \frac{1}{\cos B}.$$

The circumscribing conic

$$fyz+gzx+hxy=0,$$

that is, $$\frac{f}{x}+\frac{g}{y}+\frac{h}{z}=0,$$

goes through A' if $\quad -f+g\cos C+h\cos B=0,$

and through B' if $\quad f\cos C-g+h\cos A=0.$

These give $\quad f:g:h=\sin A:\sin B:\sin C=a:b:c,$

and the trilinear equation of the circumscribing circle is

$$ayz+bzx+cxy=0.$$

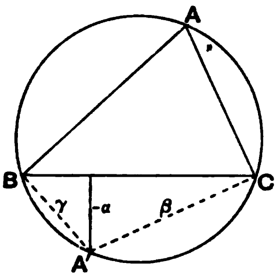

FIG. 28.

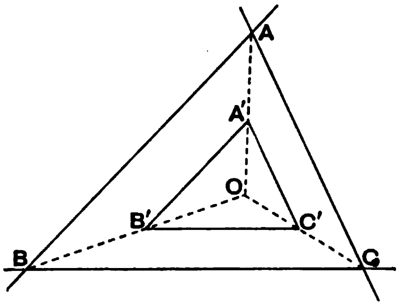

FIG. 29.

122. To find the equation of any circle, describe in it a triangle having its sides parallel to those of the triangle of reference; the joins of corresponding vertices of these two triangles (triangles in perspective) are concurrent in O; let the actual coordinates of O be h, k, l. Let the sides of the second triangle be a', b', c', then $a':a=b':b=c':c=\lambda:1$. Let x, y, z denote coordinates (actual perpendiculars) referred to ABC; x', y', z' coordinates referred to $A'B'C'$ (Fig. 29).

The equation of the circle through $A'B'C'$ is

$$a'y'z'+b'z'x'+c'x'y'=0,$$

that is, dividing by λ,

$$ay'z'+bz'x'+cx'y'=0.$$

Now $OB'=\lambda OB$; therefore $h'=\lambda h$, and the distance from $B'C'$ to BC, *i.e.* $h-h'$, is $h(1-\lambda)$. Hence

$$x'=x-h(1-\lambda)=x-\mu h,$$
$$y'=y-k(1-\lambda)=y-\mu k,$$
$$z'=z-l(1-\lambda)=z-\mu l, \text{ where } \mu=1-\lambda.$$

Making these substitutions, the equation of the circle $A'B'C'$ becomes

$$a(y-\mu k)(z-\mu l)+b(z-\mu l)(x-\mu h)+c(x-\mu h)(y-\mu k)=0,$$

that is,

$$ayz+bzx+cxy-\mu\{a(ly+kz)+b(hz+lx)+c(kx+hy)\}+\mu^2(akl+blh+chk)=0,$$

which may be written

$$ayz+bzx+cxy=\mu L_1+\mu^2 L_0,$$

where the suffixes of the L's denote the degree in the coordinates.

Making this equation homogeneous by means of the unit multiplier

$$\frac{ax+by+cz}{2\Delta},$$

x, y, z are no longer restricted to be actual perpendiculars; they are now *proportional* to the perpendiculars, that is, they are general trilinear coordinates, and the trilinear equation of any circle is

$$ayz+bzx+cxy=M_1 s+M_0 s^2,$$

that is, $\quad ayz+bzx+cxy=s\times$ a linear expression,

which may be written

$$ayz+bzx+cxy+(ax+by+cz)(lx+my+nz)=0.$$

This form of the equation shows that all circles meet infinity in the same two points, viz., those given by

$$ax+by+cz=0,$$
$$ayz+bzx+cxy=0.$$

Solving these equations, the points are found to be $(-1, e^{iC}, e^{-iB})$, $(-1, e^{-iC}, e^{iB})$ as before.

123. Comparing with this result that obtained in rectangular Cartesians, viz., that any circle is

$$x^2+y^2+gx+fy+c=0,$$

it is seen that this amounts to

$$x^2+y^2+z(gx+fy+cz)=0,$$

where $z=0$ is the line infinity. Hence the result is the same; all circles have two points in common at infinity, and the lines joining the origin to these points are

$$x^2+y^2=0.$$

124. Since the equations of any two circles can be thrown into the form

$$S=0, \quad S+sv=0,$$

one of the three line-pairs determined by their four intersections is $sv=0$; $s=0$ joins ω, ω', hence $v=0$ joins the finite intersections σ, σ', these being real or else conjugate imaginary points. The line v is certainly real, hence any two circles have a real chord of intersection which is distinct from the line infinity unless the circles are concentric. This real line v is called the radical axis of the circles.

In a pencil of conics determined by two circles, since all the conics pass through ω, ω', all are circles; and since all pass through σ, σ', the line v is the radical axis for every pair of circles in the pencil; on this account the circles are called coaxal. The properties of a coaxal system are at once derived from the investigation in §§ 79-82 on pencils of conics. There is a common self-conjugate triangle of which one vertex, A, is at sv, and therefore at infinity. The

line BC, being the polar of A with respect to every circle, is a diameter of every circle; the centres of the circles of the pencil are consequently collinear; and the harmonic properties of the complete quadrangle $\sigma\sigma'\omega\omega'$ show that this line of centres, BC, bisects $\sigma\sigma'$ (§ 39), and is perpendicular to $\sigma\sigma'$ (§ 115); this line BC is in any case real (§ 51). If $\sigma\sigma'$ be real, the points B, C are imaginary; but if $\sigma\sigma'$ be imaginary, the pencil is as represented in Fig. 24 (b), but with the conics all circles; the points B, C are real, and the circles are in two nests about B, C. Now the line-pair $B\omega$, $B\omega'$ is a conic of the pencil, and is a circle; but it is a degenerate circle; being degenerate in point coordinates, it must be regarded as an infinitely small circle (compare §§ 48, 117); it is the innermost circle of the nest about B; similarly the nest about C is limited by the degenerate circle $C\omega$, $C\omega'$. These two points B, C, which are the limits of the system, are called the *limiting points*. (Poncelet. See Salmon's *Conic Sections*, §§ 109-112.) Since they are vertices of the self-conjugate triangle, the polar of B with regard to any circle of the system is CA, and the polar of C is AB; since A is at infinity on the radical axis v, this may be stated:—The polar of either limiting point with respect to any circle of the system is a line through the other limiting point, parallel to the radical axis.

The Rectangular Hyperbola.

125. Let the points in which the conic meets infinity be harmonic with respect to ω, ω'; the conic, if real, is a hyperbola. For the lines joining C to ω, ω' are, by § 113,

$$x^2+2xy\cos C+y^2=0;$$

let the lines joining C to the points at infinity on the conic be

$$ax^2+2hxy+by^2=0;$$

these pairs are harmonic if

$$a+b-2h\cos C=0,$$

that is, if $$4h^2\cos^2 C=(a+b)^2,$$

which gives $$4(h^2-ab)=4h^2\sin^2 C+(a-b)^2,$$

hence a, b, h being real, h^2-ab is positive, and the points at infinity on the conic are real. The conic is therefore a hyperbola; its asymptotes divide $\omega\omega'$ harmonically, and are therefore at right angles, in consequence of which the curve is called a *rectangular hyperbola*.

To be told that a conic is a rectangular hyperbola is to be given one condition; for one pair of conjugate points, ω, ω', is given.

126. In general, the condition for a rectangular hyperbola is found by expressing that the lines joining any point to the points at infinity on the conic are at right angles.

For example, the coordinates being trilinears,

$$fx^2+gy^2+hz^2=0$$

is a rectangular hyperbola if

$$f+g+h=0.$$

For the line infinity is $ax+by+cz=0$; the elimination of z gives the lines joining C to the points in which the conic cuts the line infinity. Expressing that these lines

$$x^2(c^2f+a^2h)+2xyabh+y^2(c^2g+b^2h)=0$$

are harmonic with respect to

$$x^2+2xy\cos C+y^2=0,$$

the condition is found to be

$$c^2f+a^2h+c^2g+b^2h-2abh\cos C=0,$$

that is,
$$f+g+h=0.$$

Hence any conic through the intersections of two rectangular hyperbolas is a rectangular hyperbola. For taking the common self-conjugate triangle as triangle of reference, the given hyperbolas are

$$S=fx^2+gy^2+hz^2=0,$$
$$S'=f'x^2+g'y^2+h'z^2=0,$$

with the conditions

$$f+g+h=0,\ f'+g'+h'=0.$$

Hence for the general conic of the pencil,

$$\lambda S+\lambda' S'=0,$$

the sum of the coefficients, which is

$$\lambda(f+g+h)+\lambda'(f'+g'+h'),\ =0.$$

In this pencil of conics three line-pairs are included; and since all the conics of the pencil are rectangular hyperbolas, each pair must be composed of lines at right angles. Hence calling the four points P, Q, R, S, it is seen that

PQ is perpendicular to RS,
PR is perpendicular to QS,
PS is perpendicular to QR, (Fig. 30);

considering the triangle PQR, this shows that lines through the vertices perpendicular to the opposite sides are concurrent in S; S is the orthocentre of the triangle PQR; and similarly any other of the four points is the orthocentre of the triangle

formed by the remaining three. Here we have proved two theorems :—

(i.) The lines through the vertices of a triangle perpendicular to the opposite sides are concurrent;

(ii.) If a rectangular hyperbola circumscribe a triangle, it passes through the orthocentre.

127. The centre locus for any pencil is known to be a conic; if the pencil be the one just considered, this conic is of special interest.

The line infinity is $ax+by+cz=0$. The pole of this with respect to

$$fx^2+gy^2+hz^2=0$$

is given by
$$\frac{fx}{a}=\frac{gy}{b}=\frac{hz}{c}.$$

Now $f+g+h=0$, hence the locus of the pole is

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0,$$

that is,
$$ayz+bzx+cxy=0,$$

which is the circle circumscribing the triangle of reference. Fig. 30 shows that the three points A, B, C are the "centres" for the three line-pairs PQ, RS; PR, QS; PS, QR. Let the circle cut PQ again in A'. Since some particular conic of the

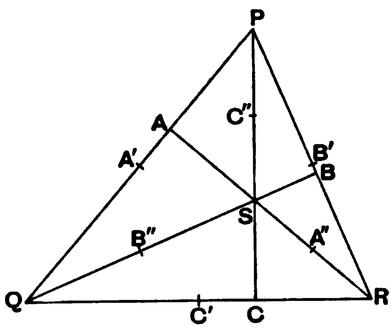

FIG. 30.

pencil has A' for its centre, A' must bisect PQ; similarly the bisections of the remaining five segments, B', C' on PR, QR; A'', B'', C'' on SR, SQ, SP, lie on the circle. Stating these relations with reference to the triangle PQR, we have the two theorems :—

(i.) The bisections of the sides of a triangle, the feet of the perpendiculars from the vertices, and the bisections of the lines joining the vertices to the orthocentre (nine points in

all) lie on a circle, called the Nine Points Circle (N.P.C.) of the triangle;

(ii.) If a rectangular hyperbola circumscribe a triangle, the locus of its centre is the N.P.C.

Note. All these theorems relate properly not to a triangle PQR, but to a particular configuration of four points $PQRS$, which may be called an orthocentric quadrangle; this is determined by two pairs of perpendicular lines. The theorems may be stated with reference to any one of the four triangles determined by the four points.

128. The equation of the N.P.C. of the triangle of reference is at once found from the fact that it passes through the bisections of the sides. These in trilinears are $\left(0, \frac{1}{b}, \frac{1}{c}\right)$ etc., hence the circle being

$$(lx+my+nz)(ax+by+cz)-2(ayz+bzx+cxy)=0,$$

l, m, n must satisfy

$$\left(\frac{m}{b}+\frac{n}{c}\right)2-2\frac{a}{bc}=0, \text{ etc.},$$

therefore the equations for l, m, n are

$$\begin{aligned} mca+nab&=a^2,\\ lbc\qquad+nab&=b^2,\\ lbc+mca\qquad&=c^2;\end{aligned}$$

these give $l, m, n=\cos A, \cos B, \cos C$, and the N.P.C. is

$$(x\cos A+y\cos B+z\cos C)(ax+by+cz)-2(ayz+bzx+cxy)=0,$$

that is,

$$(x\cos A+y\cos B+z\cos C)(x\sin A+y\sin B+z\sin C)$$
$$-2(yz\sin A+zx\sin B+xy\sin C)=0.$$

Foci.

129. Having found that there are certain exceptional lines, a question naturally presents itself with regard to any line system, viz., What exceptional lines are included in the system? A curve being given, this regarded as an envelope gives a singly infinite system of lines; we have to determine how many of these are isotropic, and how they are situated.

The class of the curve being n, there are n tangents from ω and n from ω'; these are all imaginary, but they are in conjugate pairs, and have therefore n real intersections, $n(n-1)$ imaginary intersections. Though these exceptional line elements belonging to the system cannot be directly represented in the diagram, being imaginary, yet they can be exactly

indicated by means of their real intersections. Similarly the exceptional point elements of a curve cannot be marked on the diagram, for being at infinity they are beyond the limits; but they can be indicated by means of the asymptotes. The points thus used to mark the exceptional line elements belonging to a system are called foci; in general *a focus of a curve is the intersection of an ω-tangent and an ω'-tangent;* that is, of two isotropic tangents (Plücker). It is not necessary that the tangents be conjugate, the name focus is properly applied to the $n(n-1)$ imaginary intersections as well as to the n real intersections; but practically the former are very rarely taken into account (§§ 131, 270).

130. Let the curve be a conic; the four isotropic tangents determine a complete quadrilateral. This has one pair of real vertices, ρ, ρ', and two pairs of conjugate imaginary vertices, σ, σ', ω, ω'. The lines $\rho\rho'$, $\sigma\sigma'$, $\omega\omega'$ are real, and form a self-conjugate triangle; the intersection O of $\rho\rho'$, $\sigma\sigma'$, is consequently the pole of $\omega\omega'$, that is, O is the centre of the conic, and $\rho\rho'$, $\sigma\sigma'$ are conjugate diameters. The harmonic properties of the complete quadrilateral show that $\rho\rho'$, $\sigma\sigma'$ divide $\omega\omega'$ harmonically, and are therefore at right angles; hence *the conic has a pair of conjugate diameters at right angles; these are called the axes.*

Note. If the conic be a parabola, the conception of conjugate diameters is not applicable; if it be a circle, every pair of conjugate diameters is harmonic with respect to $\omega\omega'$, and therefore at right angles; hence any pair of conjugate diameters may be regarded as axes. These two conics are excluded from the present discussion.

The real foci, ρ, ρ', are on one axis, and the harmonic properties of the quadrilateral show that $\rho\rho'$ is bisected at O; the imaginary foci σ, σ' are on the other axis, and $\sigma\sigma'$ is bisected at O.

The conception of foci here presented agrees with the definition adopted in Cartesians. For the Cartesian equation of the locus of a point P whose distance from a fixed point, the origin, is in a constant ratio to its distance from a fixed line,

$$lx+my+n=0,$$

is

$$x^2+y^2=k(lx+my+n)^2;$$

but this simply expresses that the isotropic lines $x \pm iy=0$ are tangents, and that $lx+my+n=0$ is their chord of contact. Adopting the general definition of a focus given in § 129, the directrix is defined for a conic as the polar of a focus, and the Cartesian equation follows at once.

131. A curve of class n has ordinarily n^2 foci, of which n are real. If the curve be *circular*, that is, if it pass through ω, ω', the tangent at either of these points takes the place of two of the tangents from the point; hence four foci, two real and two imaginary, coincide to form what is properly a quadruple focus, though it is frequently called a double focus, being the representative of two real foci. Thus a circle has no focus distinct from its centre.

Again, if the curve have a *parabolic* branch, that is, a branch having contact with the line infinity, the number of foci is affected. Let the point of contact be τ; the line infinity counts as an ω-tangent and as an ω'-tangent, whose intersection is at τ; thus one of the intersections of conjugate isotropic tangents is at τ. Also a certain number of the intersections of non-conjugate isotropic tangents are now at ω, ω'. For the line infinity, quà ω'-tangent, meets the $n-1$ remaining ω-tangents at ω. Thus the foci ordinarily enumerated as n real, $n(n-1)$ imaginary are now $n-1$ real; one real at τ; $2(n-1)$ imaginary at ω, ω', $(n-2)(n-1)$ imaginary not at ω, ω'. The four foci of a conic, for instance, when the conic becomes a parabola, are

one real, in the finite part of the plane;
one real, the point of contact with infinity;
two imaginary, one each at ω, ω' (not counted as foci).

There is here a difference of opinion as to the number of foci to be counted. The foci are to be distinct from ω, ω', that is agreed; but the divergence of opinion is as to the way of regarding those that come at infinity, but not at ω, ω'. Has the parabola *one* focus, or has it *two* foci, one being at infinity? Either view expresses the truth, when properly understood. Professor Cayley's view is that only those isotropic tangents are to be taken into account that are distinct from the line infinity; hence if there be contact with the line infinity, there are $n-1$ pairs of conjugate isotropic tangents to be considered, giving $(n-1)^2$ foci, of which $n-1$ are real. Adopting this view, the parabola has only one focus. A higher degree of specialization in the relation of the curve to the line infinity or to the circular points interferes still further with the number of foci. (Salmon, *Higher Plane Curves*, § 138).

132. The line equation of a conic touching the lines $\rho\omega, \rho\omega'$, $\rho'\omega, \rho'\omega'$ is known to be

$$\rho\rho' = k\omega\omega';$$

this is therefore the equation of a conic having ρ, ρ' as foci. Recalling the significance of an expression ρ, linear in ξ, η, ζ

(viz., a multiple of the distance from the point $\rho=0$ to the line ξ, η, ζ), and bearing in mind that $\omega\omega'$ is a constant, we see that this equation simply states a theorem :—

The product of the perpendiculars from the foci of a conic to any tangent is constant.

133. Plücker's conception of foci affords simple proofs of the focal properties of conics, reducing these to depend on poles and polars.

Ex. 1. Let F, T be conjugate points. FY, FZ, TP tangents; then by poles and polars (Fig. 31 (a)), FT, FP are harmonic with respect to FY, FZ. Referring this to Fig. 31 (b), let F be a focus, then FY, FZ

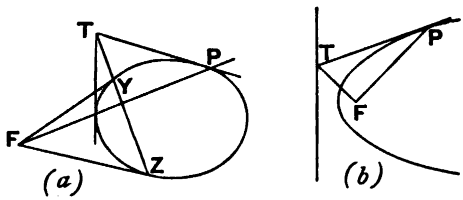

Fig. 31.

are the isotropic lines through F; FT, FP, harmonic with respect to these are therefore perpendicular. The line TYZ is the polar of F, therefore in (b) it is the directrix; and the theorem becomes :—

The part of a tangent between the point of contact and the directrix subtends a right angle at the focus.

Ex. 2. Let the polars of two points F, T meet in L, then FT is the polar of L (Fig. 32); FT, FL are harmonic with respect to FY, FZ. Placing F at a focus, T at infinity, the line YZL is the directrix, and

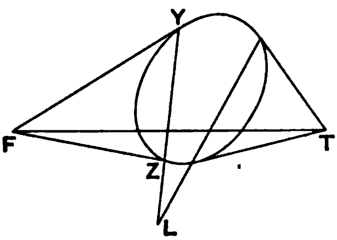

Fig. 32.

the polar of T becomes the diameter bisecting chords through T, *i.e.* bisecting chords parallel to FT; FT, FL become perpendicular lines.

Hence the locus of the bisections of parallel chords of a conic is a straight line meeting the directrix where it is met by a line through the focus perpendicular to the chords.

134. To be given a focus of a conic is a double condition, for two tangents are given. Hence the two foci amount to four simple conditions, and one point or tangent deter-

mines the conic; one point determines the conic as one of two, one tangent determines it uniquely. A system of conics with the same foci, that is, a system of confocals, is what was called in § 61 a range; the conics are all in the same imaginary quadrilateral. The investigation of §§ 79-82 applies, hence confocals fall into two "nests"; no two members of a set have any real intersections (Fig. 25 (*b*)), but two from different sets have four real intersections. All the members of a set meet the line $\omega\omega'$ in points of the same nature, and the nature is different for the two sets; that is, all of one set are ellipses, and all of the other set are hyperbolas. Hence given the foci, through any point two conics can be drawn, an ellipse and a hyperbola; and in connection with Desargues' Theorem it will be shown that these are orthogonal (§ 184).

135. *Ex.* If the centre of the inscribed conic describe a fixed line, the foci describe a cubic circumscribing the triangle of reference.

Using trilinears and the associated line system, the line equation of an inscribed conic is

$$f\eta\zeta+g\zeta\xi+h\xi\eta=0\text{..............................(1).}$$

The pole with respect to this of the line $ax+by+cz=0$, that is, of a, b, c, is

$$(gc+hb)\xi+(ha+fc)\eta+(fb+ga)\zeta=0,$$

that is, $\qquad gc+hb,\quad ha+fc,\quad fb+ga.$

Writing the equation of the straight line described by the centre in the form

$$lax+mby+ncz=0,$$

the substitution of the coordinates just found gives the condition

$$la(gc+hb)+mb(ha+fc)+nc(fb+ga)=0,$$

that is, $\qquad \dfrac{f}{a}(m+n)+\dfrac{g}{b}(n+l)+\dfrac{h}{c}(l+m)=0,$

which may be written

$$fl'+gm'+hn'=0\text{..............................(2).}$$

If a pair of foci be ρ, ρ', the line equation of the conic is

$$\omega\omega'=k\rho\rho',$$

which, if $\qquad \rho=\lambda\xi+\mu\eta+\nu\zeta,\quad \rho'=\lambda'\xi+\mu'\eta+\nu'\zeta, \qquad$ is

$$\xi^2+\eta^2+\zeta^2-2\eta\zeta\cos A-2\zeta\xi\cos B-2\xi\eta\cos C-(\lambda\xi+\mu\eta+\nu\zeta)(\lambda'\xi+\mu'\eta+\nu'\zeta)=0.$$

Comparing this with (1), which is the known line equation for the conic, we find

$$\lambda\lambda'=1,\quad \mu\mu'=1,\quad \nu\nu'=1;$$

that is, if one focus of the inscribed conic be λ, μ, ν, the other is

$$\frac{1}{\lambda},\ \frac{1}{\mu},\ \frac{1}{\nu};$$

and $\qquad \dfrac{\mu}{\nu}+\dfrac{\nu}{\mu}+2\cos A=f,$ etc.;

therefore, taking x, y, z for either focus,

$$f:g:h=\frac{y^2+2yz\cos A+z^2}{yz}:\frac{z^2+2zx\cos B+x^2}{zx}:\frac{x^2+2xy\cos C+y^2}{xy},$$

whence

$$f:g:h=x(y^2+2yz\cos A+z^2):y(z^2+2zx\cos B+x^2):z(x^2+2xy\cos C+y^2).$$

It has been found that f, g, h satisfy a linear relation (2); hence substituting these values for f, g, h, the locus of any focus of the conic (that is, the locus of the foci) is found to be

$$l'x(y^2+2yz\cos A+z^2)+m'y(z^2+2zx\cos B+x^2)+n'z(x^2+2xy\cos C+y^2)=0,$$

a cubic circumscribing the triangle of reference.

Examples.

1. Find (a) the point equation, (b) the line equation, of the circle with respect to which the triangle of reference is self-conjugate.

2. Hence show that the point equation of every circle can be thrown into the form

$$x^2\sin 2A+y^2\sin 2B+z^2\sin 2C$$
$$+(x\sin A+y\sin B+z\sin C)(lx+my+nz)=0.$$

3. Show that the equation of the N.P.C. can be written

$$x^2\sin 2A+y^2\sin 2B+z^2\sin 2C-2(yz\sin A+zx\sin B+xy\sin C)=0.$$

4. Find the line equation and the point equation of the inscribed circle.

5. Find the line equation of the circumscribing circle.

6. Show that the N.P.C., the circumscribing circle, and the circle with respect to which the triangle of reference is self-conjugate, have a common radical axis.

7. The three radical axes of three circles taken in pairs are concurrent.

8. The polars of a fixed point with regard to a coaxal system are concurrent.

How are the poles of a fixed line arranged?

9. State with regard to a confocal system the results corresponding to those in Ex. 8.

10. Every conic that passes through all the foci of a conic is a rectangular hyperbola.

11. Show how to determine the foci of a conic in trilinears.

12. What sort of (a) pencil, (b) range, is determined by two conics, these being both

(1) circles, (2) parabolas, (3) rectangular hyperbolas?

13. How many (1) circles, (2) parabolas, (3) rectangular hyperbolas, must be contained in

(*a*) a pencil, (*b*) a range?

What is the effect, in every case, of there being *more* than this necessary number?

14. If the circle circumscribing the triangle formed by three tangents to a conic pass through a focus, the conic is a parabola.

15. Find the equation of the line infinity if the quadrangle $1, \pm 1, \pm 1$, be orthocentric. Also the equation if $f, \pm g, \pm h$, be orthocentric.

16. Find the envelope of the asymptotes of a rectangular hyperbola through three fixed points.

CHAPTER VIII.

UNICURSAL CURVES. TRACING OF CURVES.

136. The parametric expression of the coordinates of a point on a curve depends on the fact that considering the points of the curve as elements, the curve itself as the space, we are dealing with one-dimensional space, this being selected out of the general two-dimensional space, the plane; that is, the statement that the point lies on the curve destroys one degree of freedom. Thus for example in polar coordinates, r being given, the point is restricted to lie on a certain circle whose centre is the origin; it is limited to this one-dimensional space; and its position in this space is determined by one more coordinate, for example, the vectorial angle θ.

Thus the two general independent coordinates of a point are involved in the two statements "the point lies on a certain curve," "the point has a particular position on this curve"; hence the homogeneous coordinates x, y, z and these implied coordinates must be expressible in terms of one another. *Must be expressible;* that is, theoretically; the two sets of quantities depend on one another, but it does not follow that this dependence can be actually expressed with any convenience or simplicity.

137. Now suppose that x, y, z are expressed in terms of the implied coordinates ϕ, μ, where $\phi=$constant limits the point to lie on the particular curve, and μ determines its position on that curve; *e.g.* taking the above example,

$$x=r\cos\theta,\quad y=r\sin\theta,$$

$r=a$ limits the point to lie on a particular circle. If then the point be limited to a circle of radius a, whose centre is the origin, the coordinates of the point are

$$x=a\cos\theta,\quad y=a\sin\theta;$$

they are expressed in terms of θ, the one coordinate required

in the one-dimensional space to which the point is limited; the constraint by which the point is kept to the curve is exhibited by the form of the expressions of x, y in terms of θ.

In the more general case, the relations being

$$x : y : z = F_1(\phi, \mu) : F_2(\phi, \mu) : F_3(\phi, \mu),$$

so long as ϕ, μ are susceptible of any arbitrary values, the point has two degrees of freedom, it can range all over the plane. But if a constant value, a, be given to ϕ, one degree of freedom is destroyed, the point is confined to the curve $\phi = a$, its position on that curve is determined by the one coordinate μ; thus the two coordinates of the point are involved in the form of the expressions for the coordinates, and the value of the parameter μ;

$$x : y : z = f_1(\mu) : f_2(\mu) : f_3(\mu).$$

The elimination of μ from these equations leaves the position of the point on the curve undetermined, but does not interfere with the fact that the point is on the curve; that is, the elimination of μ gives the equation of the curve. Thus, for example, the elimination of θ from the equations $x = a\cos\theta$, $y = a\sin\theta$, gives $x^2 + y^2 = a^2$, the equation of the circle to which the point is confined by the *form* of the expressions for x, y in terms of θ.

Unicursal Curves.

138. We now consider exclusively the case when f_1, f_2, f_3 are rational integral algebraic expressions in μ. Let the degree be not greater than n; then

$$x : y : z = a_0\mu^n + a_1\mu^{n-1} + \ldots + a_n : b_0\mu^n + \ldots + b_n : c_0\mu^n + \ldots + c_n,$$

or writing these in the homogeneous form,

$$\begin{aligned} x &= a_0\mu^n + a_1\lambda\mu^{n-1} + \ldots + a_n\lambda^n, \\ y &= b_0\mu^n + b_1\lambda\mu^{n-1} + \ldots + b_n\lambda^n, \\ z &= c_0\mu^n + c_1\lambda\mu^{n-1} + \ldots + c_n\lambda^n. \end{aligned}$$

We thus discriminate a special family of curves, characterized by the property that the coordinates of any point can be expressed rationally in terms of a single parameter μ; any such curve is said to be unicursal. Similarly a curve may be unicursal quà envelope. (Compare § 140).

In considering the intersections of two curves, one of which is unicursal, the method explained in § 90 is applicable; x, y, z for a point on the one curve are expressed in terms of μ; for a point on the other curve they are connected by an equation $F_m(x, y, z) = 0$. Combining these

two, the intersections are determined by means of the values of μ given by an equation of degree mn.

The order of the unicursal curve is at once determined by means of an arbitrary line

$$fx+gy+hz=0.$$

The equation giving the values of μ at the common points is

$$(fa_0+gb_0+hc_0)\mu^n+\ldots+(fa_n+gb_n+hc_n)=0,$$

which is of degree n. Hence an arbitrary line meets the curve in precisely n points, and the locus is of order n.

Similarly if the coordinates of a line, and therefore also the equation of the line, involve an indeterminate μ, this entering in the n^{th} degree, the envelope is of class n, and is, by definition, unicursal.

The converse does not hold; it is not true in general that the coordinates of a point on a curve of order n are algebraic expressions of degree n in a single variable. We shall prove (§§ 141, 142) that for $n=1$ or 2 the curve *is* unicursal; a single example will show that the general n-ic is not unicursal.

139. One property of unicursal curves is at once apparent; all the real points of any such curve are arranged in a single series.

For the two independent coordinates $x:y:z$ being rational integral algebraic functions of μ, we obtain all points by giving μ all values. Now a_0, a_1, etc. being real, $a_0\mu^n+\ldots+a_n$ etc. cannot be imaginary for a real μ; consequently the real μ's from 0 through ∞ to 0 give a series of points, ending where it began, that is, a single circuit. This may pass through infinity, but that does not interfere with the continuous description of the circuit by a point. Thus for example an ellipse or a hyperbola equally consists of a single circuit. As to imaginary μ's, $a_0\mu^n+\ldots+a_n$ may possibly assume a real value for $\mu=a+\beta i$, but then it assumes the same value for $\mu=a-\beta i$; now there can only be a finite number of such pairs of values $\mu=a\pm\beta i$ that will make $x:y:z$ real, and these will exist only under certain conditions; that is, there may be a finite number of real intersections of imaginary branches; these are simply isolated points (acnodes, conjugate points) that cannot be included in the description of the curve by a real tracing point. Hence the unicursal curve consists of a single circuit; it is unipartite.

Note. If any of the coefficients a_0, a_1, etc. be imaginary, real values of $x:y:z$ can be given only by *special* imaginary values of μ; the curve consists entirely of a finite number of isolated points.

Now the cubic $-x^3+x-y^2=0$, which can easily be drawn by points obtained by taking a sufficient number of arbitrary values for x, is seen to consist of two parts, one included between $x-1=0$ and $x=0$, the other between $x+1=0$ and infinity; it is bipartite, and hence it is not unicursal. Thus the statement of § 138 that the general n-ic is not unicursal is proved.

But further, a unipartite curve is not necessarily unicursal. The truth of this statement is made evident by a comparison of the cubic $x^3+x-y^2=0$ (which, drawn by points, is at once seen to be unipartite) with the cubic $-x^3+x-y^2=0$, just discussed and shown not to be unicursal. Writing in the homogeneous form, the two cubics to be compared are

$$x^3+xz^2-y^2z=0, \quad -x^3+xz^2-y^2z=0.$$

If $x^3+xz^2-y^2z=0$ were unicursal, there would be expressions for x, y, z of the form

$$\begin{aligned} x &= a_0\mu^3+a_1\lambda\mu^2+a_2\lambda^2\mu+a_3\lambda^3=f_1, \\ y &= b_0\mu^3+b_1\lambda\mu^2+b_2\lambda^2\mu+b_3\lambda^3=f_2, \\ z &= c_0\mu^3+c_1\lambda\mu^2+c_2\lambda^2\mu+c_3\lambda^3=f_3; \end{aligned}$$

and the elimination of λ, μ from these would give the result

$$x^3x+z^2-y^2z=0.$$

This is a purely algebraic statement as to the expressibility of certain connected quantities in a certain form, and has no concern with the reality or otherwise of the quantities involved. Hence writing x', $\sqrt{i}y'$, iz' for x, y, z, we see that the elimination of λ, μ from

$$x'=f_1, \quad \sqrt{i}y'=f_2, \quad iz'=f_3$$

would give $\qquad x'^3+x'(iz')^2-(\sqrt{i}y')^2(iz')=0,$

that is $\qquad -x'^3+x'z'^2-y'^2z'=0,$

and thus the coordinates of a point on the curve $-x^3+x-y^2=0$ would be rational integral algebraic expressions involving a single parameter $\mu:\lambda$; this curve would therefore be unicursal. But it has been shown to be bipartite, and therefore not unicursal; hence the curve

$$x^3+x-y^2=0,$$

though unipartite, is not unicursal.

The term unipartite has reference simply to the appearance of the curve, and relates to the distinction between real and imaginary; the term unicursal relates to the algebraic law of being of the curve, and does not refer to the distinction between real and imaginary; there is this much connection

between the two, that if the unicursal curve have any real part other than isolated points (p. 131), it is composed of a single circuit; the unicursal curve is unipartite.

140. If a curve be unicursal quà locus, it is also unicursal quà envelope. For the coordinates of a point are given by

$$x:y:z=f_1(\mu):f_2(\mu):f_3(\mu),$$

and an adjacent point is

$$f_1(\mu+\delta\mu):f_2(\mu+\delta\mu):f_3(\mu+\delta\mu).$$

Hence the line joining the two is

$$\begin{vmatrix} x & y & z \\ f_1 & f_2 & f_3 \\ f_1+\delta\mu\,.\,f_1' & f_2+\delta\mu\,.\,f_2' & f_3+\delta\mu\,.\,f_3' \end{vmatrix}=0,$$

that is,

$$\begin{vmatrix} x & y & z \\ f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \end{vmatrix}=0,$$

and the coordinates of the tangent are therefore

$$\xi:\eta:\zeta=f_2f_3'-f_2'f_3:f_3f_1'-f_3'f_1:f_1f_2'-f_1'f_2,$$

that is, they are rational integral algebraic expressions in μ, and the curve, quà envelope, is unicursal.

Ex. Show that the degree to which μ occurs in the coordinates of the tangent cannot exceed $2(n-1)$, and may fall short of this number; and that hence the class of a unicursal curve of order m is not greater than $2(m-1)$, and the order of a unicursal curve of class n is not greater than $2(n-1)$.

141. Special examples of these principles are familiar. The point f_1+kf_2, g_1+kg_2, h_1+kh_2 lies on the line

$$\begin{vmatrix} x & y & z \\ f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \end{vmatrix}=0;$$

the line f_1+kf_2, g_1+kg_2, h_1+kh_2 passes through the point

$$\begin{vmatrix} \xi & \eta & \zeta \\ f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \end{vmatrix}=0;$$

the line $u+kv=0$, that is

$$(f_1+kf_2)x+(g_1+kg_2)y+(h_1+kh_2)z=0,$$

passes through the point $u=0$, $v=0$.

Moreover, the coordinates of any point on the line joining

x_1, y_1, z_1 to x_2, y_2, z_2 can be expressed in the form x_1+kx_2, etc., and similarly for lines. Thus if $n=1$, the locus or envelope is necessarily unicursal.

142. Taking the case $n=2$, the direct theorem is proved (for the general value of n) in § 138; viz., if the coordinates of a point (line) involve an indeterminate in the second degree, the locus of the point (envelope of the line) is a conic. This may be presented in a slightly different form, using equations instead of coordinates; if the equation of a line involve an indeterminate in the second degree, the envelope of the line is a conic. The direct special proof is the following (Salmon, *Conic Sections*, p. 248). Consider the line

$$\mu^2 u+2\mu w+v=0,$$

where u, v, w are linear functions of x, y, z; and find the point equation of the envelope, that is, the locus of the intersection of consecutive lines. The consecutive line is

$$(\mu+\delta\mu)^2 u+2(\mu+\delta\mu)w+v=0,$$

whence $$(2\mu\delta\mu+\delta\mu^2)u+2\delta\mu w=0,$$

and therefore $$\mu u+w=0;$$

that is, the line $\mu^2 u+2\mu w+v=0$ has contact with its envelope on $\mu u+w=0$. Eliminating μ, the required envelope is found to be the conic

$$uv=w^2.$$

Conversely, every conic is unicursal. For $uv=w^2$ is simply the equation referred to two tangents and their chord of contact. Hence the equation of any conic can be thrown into this form, and that in a doubly infinite number of ways. Writing μ for $\frac{u}{w}$, the equation gives $\mu v=w$. Hence x, y, z are determined by

$$u-\mu w=0, \qquad w-\mu v=0;$$

that is, by $$lx+l'y+l''z-\mu(nx+n'y+n''z)=0,$$
$$nx+n'y+n''z-\mu(mx+m'y+m''z)=0;$$

whence expressions for x, y, z are obtained of the form

$$(l'-\mu n')(n''-\mu m'')-(n'-\mu m')(l''-\mu n''), \text{ etc.},$$

whence $x:y:z=a_0\mu^2+a_1\mu+a_2:b_0\mu^2+b_1\mu+b_2:c_0\mu^2+c_1\mu+c_2.$

The ordinary expressions for a point on an ellipse or hyperbola by means of the eccentric angle can be exhibited in an algebraic form. For the ellipse

$$x=a\cos\phi, \qquad y=b\sin\phi.$$

Write $$\phi=2\theta, \qquad \tan\theta=m;$$

then
$$x = a\cos 2\theta = a\frac{1-\tan^2\theta}{1+\tan^2\theta} = a\frac{1-m^2}{1+m^2},$$
$$y = b\sin 2\theta = b\frac{2\tan\theta}{1+\tan^2\theta} = b\frac{2m}{1+m^2}.$$

EXAMPLES.

1. Express the coordinates of a point on the hyperbola $x = a\sec\phi$, $y = b\tan\phi$, algebraically.

2. Express x, y, z in the equation
$$l^2x^2 + m^2y^2 - n^2z^2 = 0$$
as algebraic functions of a single variable. Deduce the eccentric angle expressions for the ellipse and hyperbola.

3. Find the line equation of the envelope of $y = mx + \dfrac{a}{m}$. Find also the point equation, and the coordinates of the point of contact.

4. Folding a leaf of a book so that the corner moves along an opposite edge, the crease will envelope a parabola.

5. Apply the process of § 138 to show that a conic meets a curve of order m in $2m$ points. Hence show that the general cubic is of class 6.

6. A line of constant length slides with its extremities on two fixed lines at right angles. Express the coordinates of the line rationally in terms of a single parameter, and hence show that the curve enveloped is of class 4 and order 6.

Find the line equation of this envelope.

The Deficiency of a Curve.

143. It has been shown that a curve of assigned order (or class) may or may not be unicursal. A few words may now be added as to the conditions for this, though no proofs can be given for the general case, the question belonging properly to the theory of Higher Plane Curves, where it is proved that the necessary and sufficient condition for a curve of assigned order to be unicursal is that it have its maximum number of double points. Thus, for example, a cubic can have one dp (double point), but not more; for if it had two, the line joining them would meet the cubic in four points. It is shown in § 101 that a quartic can have three dps; if a quartic have four dps, A, B, C, D, take these four points and any other point P

on the curve as determining points for a conic; this meets the quartic in exactly eight points. But it meets the quartic in two points at A, B, C, D, and in one point at P, therefore in nine points, which is impossible. Consequently the assumption as to the possibility of four dps on a quartic is incorrect; a quartic cannot have more than three dps. We shall show by examples how to express algebraically in terms of a single variable the coordinates of a point on a cubic with one dp or on a quartic with three dps. Note that some of the dps may come together; thus $x^4+x^2y-y^3=0$ is a quartic having at the origin a triple point caused by the crossing of three branches, tangent to $y=0$, $x+y=0$, $x-y=0$. Geometrically this is equivalent to three crossings of branches, and therefore to three dps; and it uses up all the dps that the quartic can have, for a line joining the triple point to a double point would meet the quartic in five points.

The number by which the actual number of dps (nodes or cusps, separate or in composition) possessed by a given curve falls short of the possible maximum for curves of that order is called the *deficiency* of the curve; and the theorem above stated without proof is that the necessary and sufficient condition for a curve to be unicursal is that the deficiency be zero. It is proved in § 140 that a curve if unicursal quà locus is unicursal quà envelope; combining these two, the conclusion is :—*If the point deficiency be zero, the line deficiency is also zero*; that is, if a curve have all the double points possible for a curve of that order, it has also all the double lines possible for a curve of that class.

Note. This does not mean that the presence of the double points causes the double lines to appear, for the effect is exactly the reverse; a curve of any specified order with a double point has fewer double lines than a curve of that order without a double point. But the presence of the maximum number of double points on a curve of any specified order reduces the class to such an extent that the possible number of double lines is thereby diminished and made the same as the actual number. Similarly if the class be given the occurrence of the maximum number of double lines reduces the order to such an extent that the possible number of double points is made the same as the actual number.

Thus we have found that the non-singular cubic is of class 6 (Ex. 5, § 142); now a curve of order 6 can have a certain number of double points, and a curve of class 6 can have a certain number of double lines. But if the cubic be the one considered in § 68, Ex. 2, and again in § 69, that is, one with a cusp and an inflexion, the reciprocal is of class 3. The cubic has point deficiency zero, for it has one dp, the cusp; and its line deficiency is also zero, for being only of class 3, it can have only one double line, and that presents itself as the inflexional tangent.

This theorem is a particular case of the more general theorem that for any curve the point deficiency and the line deficiency are the same; and this again is but a special case of a much more general theorem (§ 288). These theorems are here referred to,—though the proofs, depending on the theory of Higher Plane Curves, cannot be given,—for the sake of drawing attention to the importance of the conception of the deficiency, a characteristic number p which applies equally * to the point system and the line system derived from the curve of order m and class n. Curves may be classified by order, class, and deficiency $(m, n; p)$; and the special family of curves considered in § 138 and the following sections is the family $p=0$.† For cubics $p=0$ or 1; for quartics, $p=0, 1, 2$, or 3.

144. As an example, consider the cubic

$$x^3+y^3-3xy=0,$$

which in the homogeneous form is

$$x^3+y^3-3xyz=0.$$

This has a dp at xy; take a line through xy,

$$x=\lambda y;$$

of the three intersections of this and the cubic we know *two*, for two come at the dp. Hence the third must be given by a linear equation. The equation for intersections is

$$(\lambda^3+1)y^3-3\lambda y^2z=0,$$

that is, $y^2=0$, referring to the two intersections at xy,

and $$(\lambda^3+1)y-3\lambda z=0.$$

Hence for the variable intersection

$$x:y=\lambda:1, \text{ and } y:z=3\lambda:\lambda^3+1,$$

therefore $$x:y:z=3\lambda^2:3\lambda:\lambda^3+1;$$

and returning to Cartesians,

$$x=\frac{3\lambda^2}{\lambda^3+1}, \quad y=\frac{3\lambda}{\lambda^3+1}.$$

Ex. 1. Find the coordinates of the tangent in terms of λ; hence show that this cubic is of class 4, and find the reciprocal equation.

From the fact that the line deficiency is the same as the point deficiency, show that this cubic must have three inflexional tangents.

Ex. 2. Express the coordinates of a point on the quartic

$$x^4+x^2y-y^3=0$$

by the same process; determine the coordinates of the tangent, and hence find the class of this quartic.

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* See Clifford, Synthetic Proof of Miquel's Theorem; *Mathematical Papers*, pp. 39-41.

† Curves of deficiency 0 are of "genre $p=0$," "Geschlecht $p=0$."

As another example, consider the tricuspidal quartic, which by a proper choice of coordinates can be written

$$y^2z^2+z^2x^2+x^2y^2-2x^2yz-2xy^2z-2xyz^2=0,$$

or more conveniently

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}-\frac{2}{yz}-\frac{2}{zx}-\frac{2}{xy}=0.$$

Comparing this with the conic

$$X^2+Y^2+Z^2-2YZ-2ZX-2XY=0,$$

it is seen that the expression of X, Y, Z in terms of a single parameter μ gives the desired expression of x, y, z, by means of the relations

$$x,\ y,\ z=\frac{1}{X},\ \frac{1}{Y},\ \frac{1}{Z}.$$

This conic is $\quad (4X+Y-Z)^2=XY;$

and writing $\quad 2\mu X=X+Y-Z,$

we have $\quad \mu(X+Y-Z)=2Y,$

therefore $\quad X:Y:Z=1:\mu^2:(1-\mu)^2.$

Hence $\quad x:y:z=1:\dfrac{1}{\mu^2}:\dfrac{1}{(1-\mu)^2},$

that is, $\quad x:y:z=\mu^2(1-\mu)^2:(1-\mu)^2:\mu^2.$

Ex. 3. Show that the tricuspidal quartic is of class 3, and find its line equation.

145. The special process here applied to the quartic is not universally applicable. The general process is analogous to that adopted for the cubic, where a pencil of lines through the double point was used. In the case of the quartic, a pencil of conics is required; the three dps give three base points, and for the fourth any point whatever on the quartic may be taken; then of the eight intersections of the quartic and a conic of the pencil, seven are known; consequently there must be a linear equation giving the remaining intersection in terms of the parameter of the pencil.

For the tricuspidal quartic, we take the fourth point adjacent to A, that is, we take the conics of the pencil to touch the tangent to the quartic at A. This tangent is $y-z=0$; hence the equation of any conic is

$$(y-z)(ax+by+c)+y^2=0,$$

with the conditions that $x=0$ must give $yz=0$, since the conic is to go through B, C; hence $b+1=0$, $c=0$, and the conic is

$$(y-z)(ax-y)+y^2=0,$$

that is, $\quad x(y-z)=\lambda yz \quad \text{.........................}(1).$

The intersections of this and the quartic

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}-\frac{2}{yz}-\frac{2}{zx}-\frac{2}{xy}=0,$$

which can be written

$$\frac{1}{x^2}+\frac{(y-z)^2}{y^2z^2}-\frac{2}{x}\left(\frac{1}{y}+\frac{1}{z}\right)=0,$$

are given by $$\frac{y^2z^2+\lambda^2y^2z^2}{x^2y^2z^2}-\frac{2}{x}\cdot\frac{y+z}{yz}=0,$$

that is, by $$(1+\lambda^2)yz=2x(y+z) \quad\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

(1) and (2) give

$$(1+\lambda)^2yz=4xy, \qquad (1-\lambda)^2yz=4xz,$$

therefore $$x:y:z=\frac{1}{4}:\frac{1}{(1-\lambda)^2}:\frac{1}{(1+\lambda)^2}$$
$$=(1-\lambda^2)^2:4(1+\lambda)^2:4(1-\lambda)^2,$$

which can be reduced to the form first found by writing $1-2\mu$ for λ.

Curve-tracing in Homogeneous Point Coordinates.

146. When we wish to trace a curve whose equation is given in Cartesian coordinates, we determine the points in which it meets the axes, and the shape at the origin if this be a point on the curve, not because these points are specially important, but because their determination involves less algebraic work than for any other points. Here one advantage of homogeneous coordinates shows itself; we have three lines instead of two, and we have three points, any one of which may lie on the curve; hence the form of the equation gives more information than in Cartesians. On the other hand there is the disadvantage that homogeneous coordinates are not well adapted to the measurement of actual lengths, though this can be overcome by the process of § 46.

Ex. 1. $$x^4-xy^2z-y^2z^2=0.$$

Since $x=0$ gives $y^2z^2=0$, the curve passes through B and C.

Every term of the equation is of the second degree in x, z combined, therefore there is a dp at xz, *i.e.* at B; and similarly there is a dp at C.

Since $y=0$ gives $x^4=0$, all four points on AC are at C; and similarly all four points on AB are at B.

To determine the shape at C, we have to deal with points in the immediate vicinity of C, hence x, y are infinitesimal; z is finite, subject to changes which are infinitesimal and therefore negligible in comparison with z, hence z may be regarded as a constant. Since no attempt is made to determine actual lengths in finding the shape of a curve at a point, we may take any value we please for this constant; we there-

fore write $z=1$; and the terms that give the shape at C are, exactly as in Cartesians, those of lowest order in

$$x^4-xy^2-y^2=0.$$

The ordinary process shows that these are x^4-y^2, and that therefore the curve approximates to $x^4-y^2=0$; hence there are two branches $x^2 \pm y=0$, forming a tacnode (§ 69).

Similarly to find the shape at B, put $y=1$, and select terms from $x^4-xz-z^2=0$. The two tangents at the origin are $z=0$, $x+z=0$; and the shape of the branch that touches $z=0$ is given by $x^3-z=0$. The shape of the other branch may be found by continuing the expansion $z=-x$; but we can see at once that there must be inflexional contact, for $x+z=0$ meets the curve in four points at B; one is accounted for by cutting the other branch; three are to be accounted for by contact with its own branch. Moreover,

$$z(x+z)=x^4,$$

hence z and $x+z$ must have the same sign in the immediate neighbourhood of B.

Considering the curve as a whole,

$$y^2z(x+z)=x^4,$$

hence the curve can be only in certain divisions of the plane, viz., those in which z and $x+z$ have the same sign.

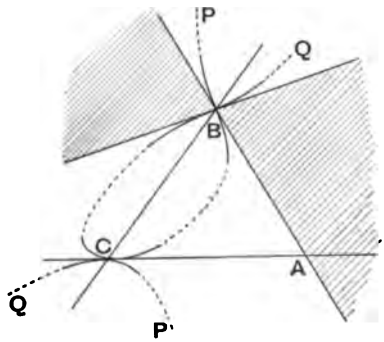

FIG. 33.

Collecting these results in a diagram, shading parts of the plane in which the curve cannot lie (Fig. 33), and noting that the curve cannot cross any of the lines

$$x=0, \quad y=0, \quad z=0, \quad x+z=0,$$

except at B, C, it is at once evident that the parts of the curve are joined as indicated by the dotted lines, the two ends P joining through infinity, and similarly for the two ends Q. If it be required to determine more exactly how this junction is effected, the asymptotes must be found by means of the equation of the line infinity.

Ex. 2. $$x^4+x^2yz-y^3z=0.$$

This passes through B, C.

For the shape at B, $y=1$, $x^4+x^2z-z=0$; hence $z=x^4$, and there is a point of undulation at B, with BA as tangent.

For the shape at C, $z=1$, $x^4+x^2y-y^3=0$; hence there is a triple point at C, the tangents being $y=0$, $x \pm y=0$.

Writing the whole equation in the form

$$zy(y^2-x^2)=x^4,$$

it is seen that so long as z is positive, $y(y^2-x^2)$ must be positive, hence the three tangents at the triple point divide the plane into regions, from the alternate ones of which the curve is excluded.

To obtain more information as to the curve, take lines through any vertex of the triangle, for example, A. Any such line $y=\mu z$ meets the curve where

$$x^4+\mu x^2z^2-\mu^3z^4=0.$$

If μ be positive, this gives two real values for $x:z$, positive and negative; if μ be negative, $=-\nu$, the equation is

$$x^4-\nu x^2z^2+\nu^3z^4=0.$$

This gives four real values for $x:z$, if $\nu^2-4\nu^3$ be positive; that is, if $\nu<\frac{1}{4}$. For $\nu=\frac{1}{4}$, the equation is $(x^2-\frac{1}{8}z^2)^2=0$; hence $y=-\frac{1}{4}z$ is a double tangent, whose points of contact are on the lines

$$z=\pm x\sqrt{8}.$$

If $\nu>\frac{1}{4}$, the four points in which the line $y=-\nu z$ meets the curve are imaginary.

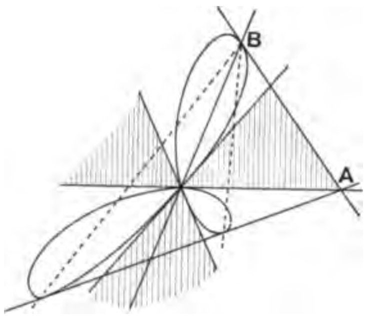

FIG. 34.

The curve is therefore as represented in Fig. 34, where as in Fig. 33 the shaded portions are those that are obviously excluded by the form of the equation. For an accurate diagram, the lines $z=-4y$, $z=\pm x\sqrt{8}$ must be inserted by the process of § 46.

Ex. 3. The curve $x(x^2-4z^2)=y(y^2-z^2)$ is easily drawn when the lines $x^2-4z^2=0$, $y^2-z^2=0$, are inserted. Drawing any line through A for $y-z=0$, $y+z=0$ is known, for $y^2-z^2=0$ are harmonic with respect to $yz=0$; and drawing any line through B for $x-2z=0$, $x+2z=0$ is known. The point $x=2z$, $y=z$ (*i.e.* the point 2, 1, 1) has here been determined, and therefore all points and lines can be marked in by § 46.

The lines already drawn divide the plane into parts, the curve being excluded from the alternate divisions, since

$$x(x-2z)(x+2z) \text{ and } y(y-z)(y+z)$$

must have the same sign.

Writing the equation in the form

$$z^2(4x-y)=x^3-y^3,$$

that is, $$z^2(4x-y)=(x-y)(x-\omega y)(x-\omega^2y),$$

we see that the curve meets $z=0$ in one real point, and that the tangent

there is $x-y=0$; and since xy is on the curve, writing $z=1$ we find that the tangent at xy is $4x-y=0$, and that this is an inflexional tangent, for it meets the curve in three points at xy.

For the sake of clearness, the line $4x-y=0$ is not indicated in Fig. 35, but all the other lines used are there shown.

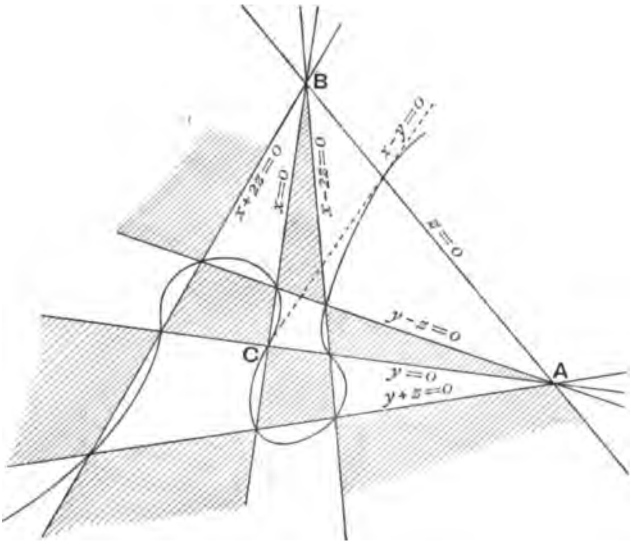

FIG. 35.

147. If an equation be given in Cartesian coordinates, it can be made homogeneous by using the line infinity for the third side of the triangle of reference. Thus, for example,

$$x^3=y^2$$

can be discussed under the form

$$x^3=y^2z,$$

a form which exhibits the inflexion at infinity on the line $y=0$, having $z=0$, that is, the line infinity, for its tangent.

But the distinct Cartesian equation

$$x^3=y,$$

written in the homogeneous form is

$$x^3=yz^2,$$

and is therefore the same, with y, z interchanged. The cusp is now at infinity, and the line infinity is the tangent there.

And finally the third equation

$$xy^2=1,$$

written in the homogeneous form being

$$xy^2=z^3,$$

is still essentially the same, though now both cusp and inflexion are at infinity, the tangents being respectively $y=0$ and $x=0$.

Similarly Ex. 1, in § 146, gives information regarding

(1) $x^4-xy^2-y^2=0$; (2) $x^4-xy-y^2=0$; (3) $1-xy^2-x^2y^2=0$.

EXAMPLES.

1. Draw the following curves, and in every case give the three Cartesian equations derived from the homogeneous form. Draw these curves referred to Cartesian axes.

(1) $x^4+x^2y^2-y^2z^2=0.$

(2) $x^4-x^3y \quad -y^2z^2=0.$

(3) $x^3-x^2z \quad -y^2z \quad =0.$

2. Express in terms of a single parameter the coordinates of a point on the following curves.

(1) $y^2=x^4-xy^2.$

(2) $x^4-xy-y^2=0.$

(3) $1-xy^2-x^2y^2=0.$

(4) $y^2=x^4+x^2y^2.$

(5) $y^2=x^2(x-1).$

3. By means of the expressions for x, y in terms of a single parameter, draw the curves in Ex. 2 by points.

4. Determine the coordinates of the tangent to the curves in Ex. 2; hence find the class of these curves.

5. Find the equation of the reciprocal to (5) in Ex. 2; draw the curve either from its equation or from the expression of the coordinates in terms of a single parameter.

CHAPTER IX.

CROSS-RATIO, HOMOGRAPHY, AND INVOLUTION.

Projection.

148. In the preceding chapters on properties of curves an algebraic classification has presented itself, depending on whether the algebraic work involved has or has not reference to the actual values of the coordinates, the properties being in the two cases *metric* and *descriptive*. The geometrical significance of this classification is now to be considered; it is found by means of the theory of Projection.

149. Given any plane figure, take any point V not in the plane; draw lines from V to all points of the given figure, and cut the conical surface so obtained by a second plane. To a point P in the first figure corresponds in the second the point P' determined by the cutting plane and the line VP; to a line AB in the first figure corresponds the line $A'B'$ determined by the cutting plane and the plane VAB; collinear points A, B, C give collinear points A', B', C'; concurrent lines a, b, c give concurrent lines a', b', c'; a curve cut by a straight line in m points gives a curve cut by a straight line in m points; and so on. Thus the second figure has a certain general resemblance to the first figure.

150. The conical surface obtained by joining the points of the given figure to V is called by the Germans the "Schein" of the figure (Reye, *Geometrie der Lage*); the second figure, the section of the "Schein" by the second plane, is what we call the "Projection" of the first figure; in German this is often called the "Schnitt"; the point V is the centre or vertex of projection. Thus a figure has any number of projections, since any point can be taken as vertex, and from this the figure can be projected on to any plane. It is at once evident that these projections may differ considerably in appearance from one another and

from the original; though the persistence of properties of collinearity and concurrence shows that a certain general resemblance will be maintained. Geometrical properties are therefore divided into projective and non-projective according as they remain unaltered or not by projection. We proceed to investigate the nature of the properties that fall under these headings.

Alteration of Magnitudes by Projection.

151. It is at once apparent that lengths of lines and magnitudes of angles (Figs. 36, 37) can be altered by projection; and yet there must be some connection, for the one figure does depend on the other.

The first thing to notice is that any segment XA can be projected so as to have any desired length. From X draw any line not in the plane (1), measure on this the desired length XA'; take any point V on the line AA', and project from V on to any plane (2) containing XA'. The projections of X, A being X, A', the segment XA becomes XA', which is of the desired length.

Note. The planes (1), (2) are not really needed here; the whole construction is in the one plane that contains $VXAA'$; in general, when the points to be projected are all in one straight line AB, the plane VAB contains the whole figure.

Evidently the degree of choice here allowed enables us to project XA, AB, two contiguous segments on a line, so as to have desired lengths. For if XA', $A'B'$ be taken of the desired lengths, the point V is the intersection of AA', BB'. . This shows that not only are the lengths of lines altered by projection, but also the ratio of these lengths is altered, for it can be made to assume any desired value; and we have to determine what it is that controls the alteration in this ratio, this being evidently not the same for all pairs of lines.

Instead of measuring XAB, $XA'B'$ from the point of intersection of the lines AB, $A'B'$, the more general question will be considered, as to the relation of $AB:BC$ to $A'B':B'C'$ (Fig. 36 (a)).

The whole diagram is in one plane, as represented in Fig. 36 (b). Draw AM, CN, $A'M'$, $C'N'$ perpendicular to VB; then

$$AB:BC=AM:NC, \quad A'B':B'C'=A'M':N'C';$$

therefore

$$\frac{AB}{BC}:\frac{A'B'}{B'C'}=\frac{AM}{NC}:\frac{A'M'}{N'C'}=\frac{AM}{A'M'}:\frac{NC}{N'C'}=\frac{VA}{VA'}:\frac{VC}{VC'}=\frac{VA}{VC}:\frac{VA'}{VC'}.$$

Thus the relation of the two ratios can be expressed in terms of the distances from V to A, C, A', C'. If now a fourth point D be taken, also on the line AC, and VD be produced to meet $A'C'$ in D',

$$\frac{AD}{DC} : \frac{A'D'}{D'C'} = \frac{VA}{VC} : \frac{VA'}{VC'},$$

therefore $$\frac{AB}{BC} : \frac{A'B'}{B'C'} = \frac{AD}{DC} : \frac{A'D'}{D'C'},$$

that is, $$\frac{AB}{BC} : \frac{AD}{DC} = \frac{A'B'}{B'C'} : \frac{A'D'}{D'C'}.$$

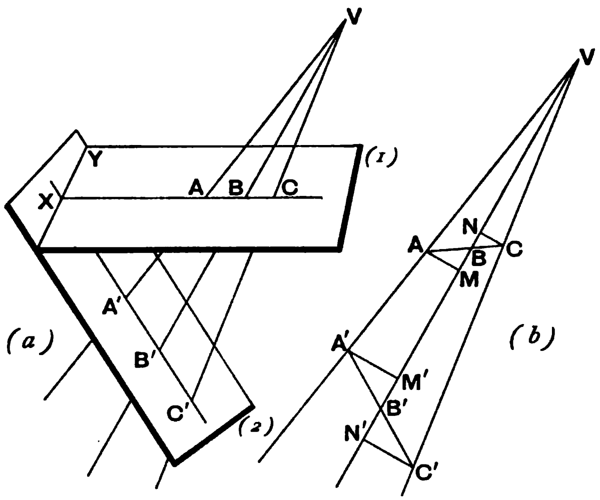

FIG. 36.

Thus although the relative position of three points on a line can be altered by projection to any desired extent, the position of any fourth point D is regulated by the fact that $\frac{AB}{BC} : \frac{AD}{DC}$ is unalterable by projection. This is a combination of lengths AB, etc. and therefore depends on metric quantities; but it is a *projective metric combination.*

152. Similarly the angles determined by concurrent lines a, b, c, can be altered to a certain extent, viz., the two angles ab, bc can be made to assume any desired magnitudes in the projection; but then the position of any fourth line d is determined.

For let a, b, c, d meet in O; their projections meet in O', the projection of O. Take any transversal $ABCD$ in the

first plane; its projection gives a transversal $A'B'C'D'$ in the second plane (Fig. 37). We know that

$$\frac{AB}{BC}:\frac{AD}{DC}=\frac{A'B'}{B'C'}:\frac{A'D'}{D'C'}.$$

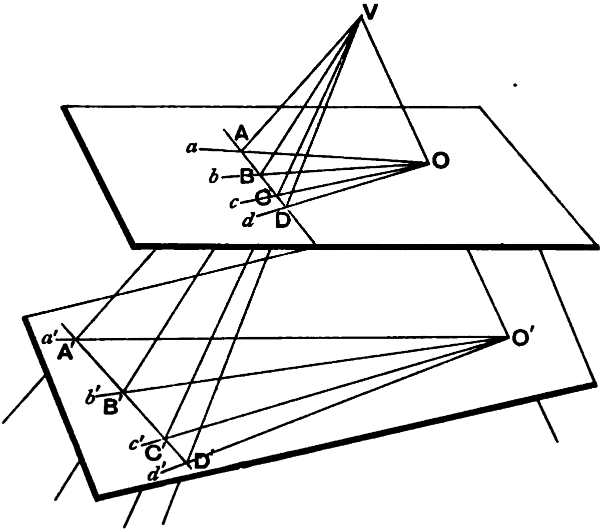

FIG. 37.

Let h be the perpendicular from O to $ABCD$. Then as in § 36

$$OA\,.\,OB\,.\sin ab=AB\,.\,h,\ \text{etc.}$$

whence
$$\frac{\sin ab}{\sin bc}:\frac{\sin ad}{\sin dc}=\frac{AB}{BC}:\frac{AD}{DC}.$$

Therefore $\frac{\sin ab}{\sin bc}:\frac{\sin ad}{\sin dc}$ is unalterable by projection; it is a *projective metric combination.*

153. It appears now that *all descriptive properties and some metric properties are projective.* Projective metric properties are those that depend on the combinations just considered. These combinations $\frac{AB}{BC}:\frac{AD}{DC}$, $\frac{\sin ab}{\sin bc}:\frac{\sin ad}{\sin dc}$ are what were defined in §§ 35, 36 as the cross-ratios of the range and of the pencil; the importance of the conception of cross-ratio consists in what has just been proved, viz., that cross-ratio is unalterable by projection. Theorems stated with reference to cross-ratio are metric theorems stated in projective form.

Compare the remark in § 40, 'Since $l:m$ does not appear in the result, it is immaterial what values the multipliers may have'; that is, cross-ratio does not depend on the nature of the coordinates.

Hence the difference in the properties of curves, first noticed in the algebraic work, is the difference between projective and non-projective; and in determining to which division to assign a property that does not appear to be purely descriptive, it is necessary to consider it from the point of view of cross-ratio.

Note. Though the conception of cross-ratio is here defined by means of metric quantities, which are combined in such a way that the metric quality is eliminated, the conception itself is descriptive, not metric, and in pure geometrical reasoning it ought to be defined accordingly. See Von Staudt, *Beiträge zur Geometrie der Lage*, pp. 131 etc.; 1856-1860.

The Group of Six Cross-ratios, Algebraically considered.

154. Since the four elements considered may be taken in any order, they afford a number of cross-ratios. It was shown in § 37 that the 24 different orders give 6 different cross-ratios, viz., $-\frac{m}{n}$, $-\frac{n}{l}$, etc., where $l+m+n=0$; that is, the different cross-ratios are

$$\begin{array}{ccc} k, & \dfrac{1}{1-k}, & 1-\dfrac{1}{k}, \\ \dfrac{1}{k}, & 1-k, & \dfrac{-k}{1-k}, \end{array}$$

where the product of members of a column is $+1$, and the product of members of a row is -1.

One set of values is inadmissible, except in special cases, viz., $k=1,\ 0,\ \infty$, for these values indicate the coincidence of some of the points considered. If for example

$$\frac{AB}{BC}:\frac{AD}{DC}=1, \text{ then } \frac{AB}{BC}=\frac{AD}{DC},$$

and therefore D coincides with B.

Again, at any rate one of the set of six, if real, is positive and less than unity. For if k be negative, $1-k$ is positive; and if $1-k$, being positive, be found greater than unity, its reciprocal which is also positive is less than unity. Let therefore k be one of the group that is positive and less than unity. The six values are now

$$\begin{array}{ccc} +<1, & +>1, & -, \\ +>1, & +<1, & -. \end{array}$$

Hence for a real value of k, there cannot be equalities between members of the same row.

But there can be equalities between members of the two rows. If then $k=1-k$, $k=\frac{1}{2}$, and the scheme of values is

$$\begin{array}{ccc} \frac{1}{2}, & 2, & -1, \\ 2, & \frac{1}{2}, & -1, \end{array}$$

accounting for all possible equalities when the cross-ratios are real. The fact that the value -1 is included in the scheme shows that the division is harmonic. For

$$\frac{AB}{BC}:\frac{AD}{DC}=-1$$

shows that $\qquad AB:BC=-AD:DC.$

If now we wish k's from the same row to be equal, the cross-ratios must be imaginary.

Let $k=\dfrac{1}{1-k}$, therefore $k^2-k+1=0$, that is, k is either imaginary cube root of -1; hence $k=-\omega$, where ω is an imaginary cube root of $+1$. The scheme now becomes

$$\begin{array}{ccc} -\omega, & -\omega, & -\omega, \\ -\omega^2, & -\omega^2, & -\omega^2. \end{array}$$

The arrangement of points that gives this scheme of cross-ratios is called equianharmonic.

155. If the four points be given by a quartic equation, it is not possible to distinguish among the six cross-ratios, for there is no way of specifying any order among the points. Hence any equation found for one of the cross-ratios will give all the others, that is, the cross-ratios will be given by a sextic equation. This sextic is most easily constructed when the conditions that the points be (i.) harmonic, (ii.) equianharmonic, are known.

(i.) It was shown in § 47 that the pairs of points given by

$$ax^2+2bx+c=0, \qquad a'x^2+2b'x+c'=0,$$

are harmonic if $\qquad ac'+a'c-2bb'=0.$

Hence the points given by

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$$

are harmonic if the expression on the left-hand side of the equation contain factors

$$x^2+2px+q, \qquad x^2+2p'x+q',$$

connected by the relation

$$q+q'-2pp'=0 \text{ ....(1).}$$

Writing $a_0=1$ for the present, and restoring it finally by

considerations of homogeneity, a comparison of coefficients gives

$$p+p'=2a_1 \quad\text{......................(2)},$$
$$4pp'+q+q'=6a_2 \quad\text{......................(3)},$$
$$pq'+p'q=2a_3 \quad\text{......................(4)},$$
$$qq'=\ a_4 \quad\text{......................(5)};$$

from (1) and (3), $$pp'=\ a_2 \quad\text{......................(6)},$$
$$q+q'=2a_2 \quad\text{......................(7)}.$$

From (2) and (6), $\qquad p=a_1\pm\sqrt{a_1^2-a_2},$

from (7) and (5), $\qquad q=a_2\pm\sqrt{a_2^2-a_4};$

substituting these in (4),

$$a_1a_2+\sqrt{a_1^2-a_2}\sqrt{a_2^2-a_4}=a_3.$$

Rationalizing, we find that the roots are harmonic if

$$a_2a_4+2a_1a_2a_3-a_3^2-a_1^2a_4-a_2^3=0.$$

Restoring a_0, the condition that the roots be harmonic is

$$g_3=0,$$

where $\qquad g_3=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$

(ii.) The roots are equianharmonic if one of the cross-ratios $=-\omega$, where $\omega^2+\omega+1=0$. Hence the condition is

$$\frac{x_1-x_2}{x_3-x_2}:\frac{x_1-x_4}{x_3-x_4}=-\omega,$$

that is, $x_1x_3+x_2x_4-x_2x_3-x_1x_4=-\omega(x_1x_3+x_2x_4-x_1x_2-x_3x_4),$

that is, $(1+\omega)(x_1x_3+x_2x_4)-\omega(x_1x_2+x_3x_4)-(x_1x_4+x_2x_3)=0,$

whence $\qquad \omega^2(x_1x_3+x_2x_4)+\omega(x_1x_2+x_3x_4)+(x_1x_4+x_2x_3)=0,$

where $\qquad \omega^2+\omega+1=0.$

The elimination of ω from these two equations gives

$$(x_1x_3+x_2x_4-x_1x_4-x_2x_3)^2$$
$$+(x_1x_2+x_3x_4-x_1x_3-x_2x_4)(x_1x_2+x_3x_4-x_1x_4-x_2x_3)=0,$$

that is, $x_1^2x_3^2+x_2^2x_4^2+x_1^2x_4^2+x_2^2x_3^2+x_1^2x_2^2+x_3^2x_4^2+6x_1x_2x_3x_4$
$$-2(x_1^2x_3x_4+x_2^2x_3x_4+x_1x_2x_3^2+x_1x_2x_4^2)$$
$$-(x_1x_2+x_3x_4)(x_1x_3+x_2x_4+x_1x_4+x_2x_3)$$
$$+x_1^2x_3x_4+x_2^2x_3x_4+x_1x_2x_3^2+x_1x_2x_4^2=0,$$

that is, $\qquad \Sigma x_h^2x_k^2+6x_1x_2x_3x_4-\Sigma x_h^2x_kx_l=0.$

Now $\quad \Sigma x_h^2x_k^2=(\Sigma x_hx_k)^2-2\Sigma x_h^2x_kx_l-6x_1x_2x_3x_4,$

therefore the condition is

$$(6a_2)^2-3\Sigma x_h^2x_kx_l=0;$$

and as $\Sigma x_h^2 x_k x_l = \Sigma x_h \Sigma x_h x_k x_l - 4x_1x_2x_3x_4 = (-4a_1)(-4a_3) - 4a_4$,
this reduces to $\quad 36a_2^2 - 48a_1a_3 + 12a_4 = 0,$
that is, to $\quad 3a_2^2 - 4a_1a_3 + a_4 = 0.$
Hence the roots are equianharmonic if

$$g_2 = 0,$$

where

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2.$$

156. Let the sextic whose roots are the six values of a group of cross-ratios be

$$\phi^6 - p\phi^5 + q\phi^4 - r\phi^3 + s\phi^2 - t\phi + u = 0.$$

If ϕ be any root, $\frac{1}{\phi}$ is also a root; therefore

$$u = 1, \qquad t = p, \qquad s = q,$$

and the equation is of the form

$$\phi^6 - p\phi^5 + q\phi^4 - r\phi^3 + q\phi^2 - p\phi + 1 = 0 \;\ldots\ldots\ldots\ldots(1).$$

Again, $1-\phi$ is a root, therefore

$$(1-\phi)^6 - p(1-\phi)^5 + q(1-\phi)^4 - r(1-\phi)^3 + q(1-\phi)^2 - p(1-\phi) + 1 = 0 \;(2)$$

is to be the same as (1). Comparing the coefficients in (1) and (2), we find that

$$p = 3, \qquad r = 2q - 5;$$

hence the equation reduces to

$$\phi^6 - 3\phi^5 + q\phi^4 + (5-2q)\phi^3 + q\phi^2 - 3\phi + 1 = 0 \;\ldots\ldots\ldots\ldots(3),$$

a form involving only the one quantity q, agreeing with what is already known, that the six cross-ratios are all determined when one is known.

But equation (3) can be written in a better form. One possible set of roots is $(-1, 2, \frac{1}{2})^2$; another is $(-\omega, -\omega^2)^3$. Hence some value of the one quantity involved must throw (3) into the form

$$\{(\phi+1)(\phi-2)(\phi-\tfrac{1}{2})\}^2 = 0,$$

that is, into

$$\{(\phi+1)(\phi-2)(2\phi-1)\}^2 = 0;$$

and some other value must give the form

$$\{(\phi+\omega)(\phi+\omega^2)\}^3 = 0;$$

hence (3) must be reducible to

$$\lambda\{(\phi+1)(\phi-2)(2\phi-1)\}^2 + \mu\{(\phi+\omega)(\phi+\omega^2)\}^3 = 0 \;\ldots.(4).$$

Comparing coefficients, we find

$$4\lambda + \mu = 1, \qquad -3\lambda + 6\mu = q,$$

hence λ and μ are linearly determinable, and the form (4)

may be adopted instead of (3). We have now to determine λ, μ so that the group of cross-ratios given by (4) shall be the group belonging to the quartic

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0.$$

Since $\mu=0$ gives the harmonic group, the vanishing of μ must imply $g_3=0$, and nothing else; hence $\mu=mg_3^j$. Again, $\lambda=0$ gives the equianharmonic group, hence the vanishing of λ must imply $g_2=0$, and nothing else; therefore $\lambda=lg_2^i$. Now g_3 is of degree 3, g_2 of degree 2 in the coefficients; equation (4) must be homogeneous in the coefficients a_0, a_1, etc., therefore

$$3j=2i;$$

hence $$i=3h,\quad j=2h,$$

and writing for the numerical multiplier $-\dfrac{m}{l}$ the single letter M, equation (4) becomes

$$g_2^{3h}\{(\phi+1)(\phi-2)(2\phi-1\}^2=Mg_3)^{2h}\{(\phi+\omega)(\phi+\omega^2)\}^3\ldots(5),$$

where $$g_2=a_0a_4-4a_1a_3+3a_2^2,$$
$$g_3=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

To determine h and M, take a special quartic, having roots

$$0,\ 1,\ 1-k,\ \infty.$$

One cross-ratio is $\dfrac{1-k-1}{0-1}:\dfrac{1-k-\infty}{0-\infty}$, that is, k.

The quartic is $$x(x-1)(x-1+k)=0,$$
that is, $$0\,.\,x^4+x^3-(2-k)x^2+(1-k)x=0.$$

Writing this as

$$0\,.\,x^4+12x^3-12(2-k)x^2+12(1-k)x=0,$$

we have $a_0=0$, $a_1=3$, $a_2=-2(2-k)$, $a_3=3(1-k)$, $a_4=0$;

therefore $$g_2=12(k^2-k+1),$$
and $$g_3=-4(k+1)(k-2)(2k-1).$$

The cross-ratio sextic is to be satisfied by $\phi=k$; hence, identically,

$$12^{3h}(k^2-k+1)^{3h}\{(k+1)(k-2)(2k-1)\}^2$$
$$=M\,.\,4^{2h}\{(k+1)(k-2)(2k-1)\}^{2h}\{(k+\omega)(k+\omega^2)\}^3,$$

that is,

$$12^{3h}(k^2-k+1)^{3h}\{(k+1)(k-2)(2k-1)\}^2$$
$$=M\,.\,4^{2h}\{(k+1)(k-2)(2k-1)\}^{2h}(k^2-k+1)^3;$$

therefore $$h=1,\quad 12^3=M\times 4^2,$$

whence $$M=4\times 27,$$

and consequently the cross-ratio sextic (5) is

$$g_2^3\{(\phi+1)(\phi-2)(2\phi-1)\}^2=4\times 27g_3^2\{(\phi+\omega)(\phi+\omega^2)\}^3,$$

that is, $$g_2^3\{(\phi+1)(\phi-2)(\phi-\tfrac{1}{2})\}^2=27g_3^2\{(\phi+\omega)(\phi+\omega^2)\}^3.$$

The Group of Six Cross-ratios, Geometrically considered.

157. The twenty-four orders for the four letters A, B, C, D were obtained in § 37 by associating the points in pairs AB, CD; AC, BD; AD, BC, thus obtaining three groups of eight; the eight different orders in a group were found to give two different cross-ratios, these being reciprocal, and thus six values were obtained. The same result might be arrived at by considering (1) the arrangements in which A stands first, six in number; (2), (3), (4), similar groups in which B, C, D occupy the first place. The six members of the first group, viz.,

(1) $ABCD$, (2) $ADBC$, (3) $ACDB$,
(1)′ $ADCB$, (2)′ $ACBD$, (3)′ $ABDC$,

give the six different values, corresponding to the scheme in § 154. Thus, in general, if we keep A in the first place, no change in the order of the remaining letters is admissible, agreeing with the rule:—*Any two letters may be interchanged if at the same time the other two be interchanged.* It may however happen that in special cases other interchanges are admissible.

I. Suppose that (1) is unaltered by the interchange of C, D. This interchange rearranges the six orders as follows:—

(3)′, (2)′, (1)′,
(3), (2), (1).

Hence we must have (3)′=(1); and therefore also (1)′=(3), and hence (2)′=(2); thus the group is now the harmonic group, and the interchangeable letters determine one of the pair of segments with regard to which the harmonic relation holds. In the case supposed, C, D being interchangeable, the segments AB, CD are harmonic.

II. Suppose that (1) is unaltered by a cyclic interchange of B, C, D, that is,

$$\{ABCD\}=\{ADBC\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}.$$

If then we interchange the letters occupying any two positions in the left-hand member of this, and make the same interchange on the right, the results will be the same;

therefore $\{ADCB\} = \{ACBD\}$ ....(ii).,

$\{ACDB\} = \{ABCD\}$ ....(iii.),

and $\{ABDC\} = \{ADCB\}$ ....(iv.).

From (i.) and (iii.), $\{ABCD\} = \{ADBC\} = \{ACDB\}$,

from (ii.) and (iv.), $\{ADCB\} = \{ACBD\} = \{ABDC\}$;

that is,

$$(1) = (2) = (3),$$

$$(1)' = (2)' = (3)'.$$

The group is now the equianharmonic group already considered; and as we have considered all possible interchanges of letters, there are no more special cases to consider.

Thus the distinguishing characteristic of the harmonic arrangement is that a single interchange is permissible; while for the equianharmonic arrangement a cyclic interchange of three letters is permissible.

Note. The symbol $\{ABCD\}$ means properly the group of six cross-ratios; now this group is the same in whatever order the points may be taken, and consequently $\{ABCD\}$ and $\{ACBD\}$ contain the same six values. But we do not write

$$\{ABCD\} = \{ACBD\},$$

unless the interchange of B, C is admissible, that is, unless the segments AD, BC are harmonic.

Homographic Ranges and Pencils.

158. The projection of a range has been shown to be an 'equal' range, the ranges being estimated quantitatively by cross-ratios.

This second range can now be moved in space, so that the two are no longer in projective position; but they remain equal. Instead of speaking of the ranges as *equal*, which suggests metric determinations, they may be spoken of in all positions as *projective*, a term which has no metric reference; a special term is then required for the ranges in projective position, and for this *perspective* is used. Thus projective ranges in space of three dimensions are in perspective when the joins of corresponding points are concurrent; and projective pencils are in perspective when the intersections of corresponding rays are collinear. (From § 152, with the help of the fact that any line and its projection meet on the line of intersection of the planes.)

Considering the limiting case, when the two planes are coincident, and the point V lies on them, this gives us ranges and pencils in perspective in a plane; but the double idea here involved is simply what can be derived by the principle of duality from the idea of ranges in perspective in a plane.

Thus there is no necessity to appeal to three-dimensional geometry for the idea of perspective, though a gain in clearness is thereby obtained.

159. The ranges must now be considered as comprising an indefinite number of points; and from the fact that the two ranges are projective it follows that the cross-ratio of any four points in the one is equal to the cross-ratio of their correspondents in the other; this is expressed by the notation

$$\{ABCDE\ldots\} = \{A'B'C'D'E'\ldots\};$$

another notation is frequently used, especially when the purely descriptive view of the subject is adopted, the ranges being regarded as projective, viz.,

$$ABCDE\ldots \barwedge A'B'C'D'E'\ldots.$$

(Reye, *Geometrie der Lage.*)

Projective ranges are in perspective if three joins of corresponding points be concurrent. For if AA', BB', CC' meet in V, the ranges are necessarily in one plane, the plane VAB; and if DD' do not pass through V, let VD meet $A'B'$ in D''. Then, by projection from V,

$$\{ABCD\} = \{A'B'C'D''\};$$

and by the given relation of the ranges,

$$\{ABCD\} = \{A'B'C'D'\};$$

therefore $\qquad \{A'B'C'D''\} = \{A'B'C'D\}'$,

whence $\qquad A'D'' : D''B' = A'D' : D'B'$,

that is, the segment $A'B'$ is divided in the same ratio by the two points D', D''; hence D'' is the same as D'. Thus the line joining any two correspondents D, D' passes through V.

A particular case of this is the theorem :—

If one pair of correspondents in two projective ranges coincide, the joins of corresponding points are concurrent; and similarly, if one pair of corresponding rays in two projective pencils coincide, the intersections of corresponding rays are collinear.

160. Projective ranges or pencils are also called homographic; that is, *two ranges or pencils are homographic when the cross-ratio of any four elements of the one is equal to the corresponding cross-ratio formed from the other.* And this definition applies also to the case of two different configurations; a range and a pencil can be homographic.

But homography can be considered from an entirely different point of view. *The one-dimensional figures we are concerned with are homographic when there is a one-one correspondence between their elements.* When we say that there is a (1, 1) correspondence between the elements of two configurations, we mean that there is a construction, geometrical or algebraic, by means of which an element of one configuration can be derived from an element of the other configuration; and that this construction is of such a nature that to one element in the one configuration there corresponds one element in the other configuration. Thus, for example, there is a (1, 1) correspondence between two ranges in perspective, and the construction is by means of lines through V. But if we take a straight line and a conic in a plane, and a point V not on either locus, though the points of the two can be connected—made to correspond—by means of lines through V, the correspondence is not (1, 1); one point on the line now gives two points on the conic, while one point on the conic leads to one point on the line; the correspondence thus instituted between the straight line and the conic is therefore a (1, 2) correspondence.

Note. This does not assert that there cannot be a (1, 1) correspondence between the line and the conic; such a correspondence can in fact be instituted, for example, by taking the point V on the conic.

161. This second definition of homography is now to be proved equivalent to the one first adopted. For simplicity, the proof is worded so as to apply to the case of two ranges; but as it depends only on the determination of the elements of the configuration by a single coordinate, it can be at once applied to the other cases.

Let the points of one range be given by their distances, x, from a fixed origin O on the line; and let y, O' belong to the second range. Then since there is a correspondence, x is a function of y, and y is a function of x. But x must be a direct function of y, for an inverse function ($\sin^{-1}y$, $\sqrt{y}$, etc.) is not one-valued; and y must be a direct function of x. The only possible way of satisfying these two conditions is to have

$$x = \text{one linear function of } y \text{ divided by another};$$

that is, $$x=\frac{py+q}{ry+s},$$

from which $$y=\frac{-sx+q}{rx-p};$$

hence the relation between x, y is of the form

$$\alpha xy+\beta x+\gamma y+\delta=0,$$

linear in x, y separately.

Let x_1, x_2, x_3, x_4 be four points of the first range, y_1, y_2, y_3, y_4 their correspondents. To show that

$$\frac{x_1-x_2}{x_2-x_3}:\frac{x_1-x_4}{x_4-x_3}=\frac{y_1-y_2}{y_2-y_3}:\frac{y_1-y_4}{y_4-y_3},$$

express $(x_1-x_2)(x_4-x_3)$ in terms of y. We have

$$\begin{aligned}x_1-x_2&=\frac{py_1+q}{ry_1+s}-\frac{py_2+q}{ry_2+s}\\&=\frac{(py_1+q)(ry_2+s)-(py_2+q)(ry_1+s)}{(ry_1+s)(ry_2+s)}\\&=\frac{(ps-qr)(y_1-y_2)}{(ry_1+s)(ry_2+s)},\end{aligned}$$

therefore

$$(x_1-x_2)(x_4-x_3)=\frac{(ps-qr)^2(y_1-y_2)(y_4-y_3)}{(ry_1+s)(ry_2+s)(ry_3+s)(ry_4+s)};$$

and similarly

$$(x_2-x_3)(x_1-x_4)=\frac{(ps-qr)^2(y_2-y_3)(y_1-y_4)}{(ry_1+s)(ry_2+s)(ry_3+s)(ry_4+s)}:$$

dividing one by the other,

$$\frac{(x_1-x_2)(x_4-x_3)}{(x_2-x_3)(x_1-x_4)}=\frac{(y_1-y_2)(y_4-y_3)}{(y_2-y_3)(y_1-y_4)};$$

that is, (1, 1) *correspondence of one-dimensional figures implies equality of corresponding cross-ratios.*

162. The correspondents to any three points can be chosen arbitrarily; but the correspondent to any other point is thereby determined.

For from the first definition,

$$\{ABCD\}=\{A'B'C'D'\};$$

hence if A', B', C' be chosen arbitrarily to correspond to A, B, C, the correspondent to D is known.

From the second definition, the two parameters x, y are connected by a relation

$$\alpha xy+\beta x+\gamma y+\delta=0;$$

here there are three quantities to be determined,

$$\alpha:\beta:\gamma:\delta,$$

and three sets of values for x, y determine these; hence any fourth y is known in terms of x.

In the linear relation between x, y the origins for the two systems are any points O, O'; these can be changed to any other points without altering the *form* of the relation, though the values of $\alpha : \beta : \gamma : \delta$ will be altered.

Homographic Systems with the same Base.

163. Since the relative position of projective ranges can be altered to any extent, the two lines on which they are marked can be supposed to coincide; and since either origin can be changed to any desired point, the same origin can be adopted for the two systems; let the parameters of the two systems be now denoted by x, x'; these are connected by a relation

$$\alpha xx' + \beta x + \gamma x' + \delta = 0.$$

At any point of the line there is a point of the first range, P, and also a point of the second range, Q'; this can be expressed by saying that every point of the line is counted twice, once for each range. Let the point of the second range that corresponds to P be P'; and similarly let Q in the first range correspond to Q' in the second. Let $OP = \lambda$.

Since $$x' = -\frac{\beta x + \delta}{\alpha x + \gamma}, \quad \text{and} \quad x = -\frac{\gamma x' + \delta}{\alpha x' + \beta},$$

therefore $$OP' = -\frac{\beta\lambda + \delta}{\alpha\lambda + \gamma}, \quad \text{and} \quad OQ = -\frac{\gamma\lambda + \delta}{\alpha\lambda + \beta};$$

and thus P' and Q do not come together; that is, the correspondent to any point of the line is different according as the point is regarded as belonging to the first range or to the second.

164. A question that naturally occurs is:—Can a point correspond to itself?

This requires $x' = x$; the equation connecting x, x' becomes

$$\alpha x^2 + (\beta + \gamma)x + \delta = 0,$$

showing that there are two such points, real or imaginary; these are the double points of the system. Whether these are real or imaginary, the point midway between them is real; if this be taken as origin,

$$\beta + \gamma = 0,$$

and the homographic relation between the two systems is expressed by

$$\alpha xx' + \beta(x - x') + \delta = 0.$$

Calling the double points F_1, F_2, we have, since each corresponds to itself,

$$\{ABC\ldots F_1F_2\}=\{A'B'C'\ldots F_1F_2\}.$$

Similarly homographic pencils with a common vertex form a system with two double lines.

Homographic Systems with different Bases.

165. From the second definition of homography it is at once evident that if u, v be elements of the same nature, as also u', v', the two systems $u+\lambda v$, $u'+\lambda v'$ are homographic. (Compare §§ 40, 41, where it is shown that the cross-ratio of the configuration $(u, v;\ u+kv, u+k'v)$ is $k:k'$.) Hence the locus of the intersection of corresponding rays of two homographic pencils is obtained by eliminating λ from the equations

$$u+\lambda v=0,\quad u'+\lambda v'=0;$$

it is therefore the conic

$$uv'-u'v=0;$$

that is, the locus of the intersection of corresponding rays of two homographic pencils is a conic through the vertices of the pencils. And by the same algebraic work, the envelope of the line joining corresponding points of two homographic ranges is a conic, touching the lines on which the ranges are marked. (Compare with Chasles' two fundamental properties, § 88.)

166. The purely geometrical proof is interesting, for it is by means of projective pencils and ranges in non-projective position that conics are introduced into descriptive geometry.

The intersections of corresponding rays of the pencils form a singly infinite series of points in a determinate order, that is, a curve; to determine the order of this curve, consider any transversal; this cuts the two pencils in projective ranges, $ABCD\ldots$, $A'B'C'D'\ldots$. Now a point in which the transversal meets the curve is an intersection of corresponding rays, and is therefore a point which coincides with its correspondent. Hence the transversal meets the curve in the double points of the system $ABC\ldots A'B'C'\ldots$; that is, in two points. The locus is therefore of the second order. (Reye, *Geometrie der Lage.*)

Involution.

167. The correspondence considered in §§ 163, 164 is between the points on a line taken in one way and the points on that line taken in another way; it is not strictly a correspondence between points of the line. For a point A, regarded as P, gives a certain correspondent P'; regarded as Q', this same point A gives a different corre-

spondent Q. A (1, 1) correspondence between points of the line would associate them in pairs AA', so that A corresponds to A', and A' to A; *when the elements of a one-dimensional space are thus associated in pairs of correspondents, they are said to form an Involution;* and any finite number of these pairs of elements are spoken of as being in involution.

Counting all the points of the line twice, as is done when two homographic ranges are marked on one line, the arrangement just defined as an involution appears to be a special case of homography; it is a (1, 1) correspondence between the two linear aggregates of points, the ordinary equation expressing the homographic relation, viz.,

$$axx'+\beta x+\gamma x'+\delta=0 \text{......................}(1),$$

being specialized so that the correspondent to a point A is the same whether A is regarded as P or as Q'. Hence (§ 163)

$$\frac{\beta\lambda+\delta}{a\lambda+\gamma} \text{ and } \frac{\gamma\lambda+\delta}{a\lambda+\beta}$$

must be the same for all values of λ. This requires $\beta=\gamma$, and (1) becomes

$$axx'+\beta(x+x')+\delta=0 \text{ ....}(2).$$

But although this reduction of involution to a special case of homography by the device of counting all points of the line (all elements of the one-dimensional space) twice is often convenient, it yet entirely disguises the real difference between the two conceptions. *Homography is a (1, 1) correspondence between the elements of two different spaces; involution is a (1, 1) correspondence between pairs of elements of one space.*

168. Applying the conclusions already obtained for homographic systems on a line to the case of involution, or working them out afresh from equation (2) of the last section, it is seen that the cross-ratio of any four points on the line is equal to that of their correspondents. Stated with reference to points A, B, and their correspondents A', B', this tells us nothing about the points. For

$$\{ABA'B'\}=\{A'B'AB\}$$

identically, whether the points are regarded as belonging to an involution or not. But if three points A, B, C and their correspondents enter into the relation it becomes

$$\{ABCA'\}=\{A'B'C'A\},$$

and thus the correspondent to C is determined when two pairs of correspondents A, A'; B, B' are known. Hence

two pairs of points, or two segments, determine an involution; three pairs of points are not in involution unless they satisfy a certain condition.

169. If in two homographic ranges on a line one point, other than a double point, can be found that has the same correspondent whether regarded as belonging to the first range or the second, then the ranges are in involution.

For if the point be λ, and its correspondent μ, where $\lambda \neq \mu$,

$$x = \lambda \text{ is to give } x' = \mu,$$

and

$$x = \mu \text{ is to give } x' = \lambda.$$

Hence from the general homographic equation, λ, μ must satisfy the two equations

$$\alpha\lambda\mu + \beta\lambda + \gamma\mu + \delta = 0,$$
$$\alpha\lambda\mu + \beta\mu + \gamma\lambda + \delta = 0:$$

from which, by subtraction,

$$(\beta - \gamma)(\lambda - \mu) = 0,$$

that is,

$$\beta = \gamma,$$

and therefore the equation is the equation expressing involution.

Any two homographic ranges can be placed so as to be in involution; and this can be done in two ways.

For let O_1 in the first correspond to infinity in the second, and let O_2 in the second correspond to infinity in the first. Bring the bases of the two ranges together, placing O_1 and O_2 together, at O. Then the point O, whether considered as belonging to the first range or the second, has the same correspondent, viz., the point at infinity; the ranges are therefore in involution; and as the required placing of the bases can be accomplished in two ways, the second part of the statement follows.

Double Elements of an Involution.

170. If an element coincide with its correspondent, there is a double element of the involution. The condition $x' = x$ reduces the equation

$$\alpha x x' + \beta(x + x') + \delta = 0$$

to

$$\alpha x^2 + 2\beta x + \delta = 0,$$

showing that there are two double elements, real or imaginary; that is, the involution of lines through a point contains two double lines, the involution of points on a line contains two double points. Dealing with this last case, the point

midway between the two double points is real; taking it as origin, $\beta=0$, and the involution is expressed by

$$axx'+\delta=0,$$

that is, by
$$xx'=k,$$

and the double points are given by

$$x^2=k.$$

This special point, the centre, corresponds to the point at infinity on the line; for $x=0$ gives $x'=\infty$. Two cases arise according as k is (i.) positive, (ii.) negative.

(i.) If k be positive, the double points are real; x, x' have the same sign, and therefore corresponding points are on the same side of the centre.

(ii.) If k be negative, the double points are imaginary; x, x' have different signs, and therefore corresponding points are on opposite sides of the centre.

Note. Imagine a point P to describe the line, then its correspondent P' also describes the line. Let P start from O, the centre, P' therefore starts from infinity; let P move in the positive direction from O through infinity to O.

In case (i.) since P and P' are on the same side of O, P' travels to meet P; the two coincide somewhere between O and infinity, at F_1; F_1 is a double point. Similarly P having passed through infinity, and simultaneously P' through O, P, P' are now to the left of O, and are travelling towards one another; they meet at F_2, the second double point.

In case (ii.) since P and P' are to be on opposite sides of O, they pursue each other along the line, but do not coincide in real points; they remain in the two distinct segments bounded by O and infinity; as P changes from the first of these to the second, P' changes from the second to the first.

Similarly in an involution of lines with real double lines, correspondents revolve in opposite directions; in an involution with imaginary double lines, correspondents revolve in the same direction, and do not overtake each other.

The double points (or lines) are sometimes called foci (or focal lines). When these are real, the involution is hyperbolic; when imaginary, the involution is elliptic. An elliptic involution is overlapping; a hyperbolic involution is non-overlapping. In Fig. 38, the involution determined on the lower transversal by the overlapping segments AA', BB' is elliptic; the involution determined on the upper transversal by the non-overlapping segments AA', BB' is hyperbolic, and in this there are segments such as CC', which is entirely contained by BB'; CC' and BB' are also non-overlapping.

171. In the involution AA', BB', let the centre be O; this corresponds to infinity, therefore

$$\{AA'BO\}=\{A'AB'\infty\},$$

that is, $$\frac{AA'}{BA'} : \frac{AO}{BO} = \frac{A'A}{B'A} : \frac{A'\infty}{B'\infty},$$

therefore $$\frac{AO}{BO} = \frac{AB'}{BA''}$$

whence O is determined.

But a construction that is practically more convenient than this equation can be obtained by means of circles. Take any point G off the line (Fig. 38) and describe circles $AA'G$, $BB'G$;

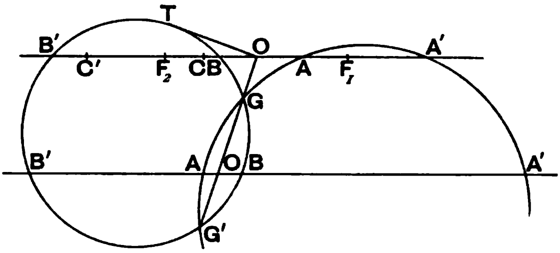

Fig. 38.

these have one real intersection G, and they have therefore another real intersection G'; let GG' meet the base in O. By the properties of circles,

$$OA \,.\, OA' = OG \,.\, OG' = OB \,.\, OB',$$

therefore O is the centre of the involution.

If now O be not between A, A', it is outside the circles GAA', GBB' (Fig. 38, upper part); draw a tangent OT, then

$$OT^2 = OB \,.\, OB' = OA \,.\, OA',$$

therefore marking off on the line

$$OF_1 = OT, \quad \text{and} \quad OF_2 = OT,$$

F_1, F_2 are the double points of the involution.

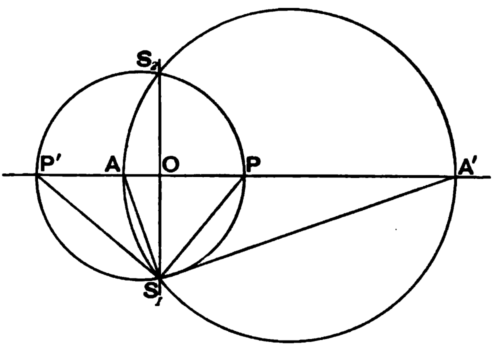

Fig. 39.

If O be between A, A', that is, inside the circle, no real

tangents can be drawn, and the double points are imaginary. In this case, describing a circle on AA' as diameter, let the double ordinate through O be S_1S_2 (Fig. 39). The two points S_1, S_2 are of service in constructions.* From the circle we have

$$OS_1 . OS_2 = OA . OA', \quad \text{and} \quad OS_1 = -OS_2,$$

therefore $$OF_1^2 = OA . OA' = -OS_1^2.$$

Note. The constructions here given, while practically convenient, are open to an important theoretical objection. The idea of an involution with its double elements is purely descriptive; and though the centre cannot be determined by purely descriptive constructions, yet it depends only on the line infinity; but these constructions depend on the circular points. A purely descriptive construction is given in § 195.

172. Any number of pairs of correspondents can be inserted when one pair and the centre are known. For if any line through O meet any circle through AA' in JJ', points P, P' on the base determined so that

$$OP = OJ, \quad OP' = OJ',$$

give $$OP . OP' = OJ . OJ' = OA . OA',$$

and are therefore correspondents in the involution. This construction is available for an involution of either kind; if however the involution be elliptic, a special construction can be used which is interesting as illustrating the relation of involution and homography. In Fig. 39,

$$PO . OP' = OS_1^2,$$

therefore PS_1P' is a right angle.

Hence if a right angle revolve about its vertex, the two legs describe an involution on any transversal; pairs of correspondents can be inserted by means of perpendicular lines through S_1.

Now suppose *any* constant angle PSP' to revolve about its fixed vertex S (Fig. 40); the two legs describe homographic ranges on any transversal, but these are not in involution.

For when the first leg passes through P', the position of the second leg is determined by taking

$$\angle P'SQ' = \angle PSP',$$

* If, however, we adopt the ordinary two-dimensional representation of imaginary values, and represent $\alpha + \beta i$ by the point α, β, the double points are represented by S_1, S_2. The whole subject of the cross-ratios of points given by a binary equation is most satisfactorily treated by means of the complex variable; the points real or imaginary can then be represented on the plane. For examples, see Harkness and Morley, *Theory of Functions*, §§ 29-45.

and Q' does not come at P unless this constant angle is a right angle; but this being so, SQ' is along PS produced, and therefore Q' is at P. In this case the legs of the constant angle describe an involution.

Note. The involution of perpendicular lines through a point is sometimes called circular; and what has just been proved shows that any elliptic involution on a line is a section of a circular involution.

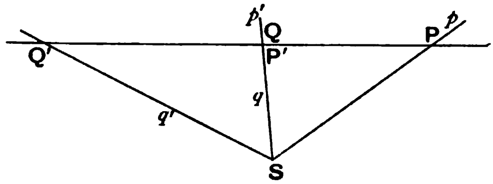

FIG. 40.

173. Since the cross-ratio of any four points is equal to that of their correspondents,

$$\{AA'F_1F_2\} = \{A'AF_1F_2\},$$

therefore the pairs AA', F_1F_2 are harmonic; that is, *the double elements of an involution are harmonic with respect to any pair of correspondents.* Hence an involution is determined by its double elements; it consists of all pairs of elements that are harmonic conjugates with respect to the double elements; for this reason the pairs of correspondents are called conjugates. Hence one pair of points can be found harmonic with respect to each of two given pairs on a line; they are the double elements of the involution determined by the given pairs, and are real or imaginary according as this involution is hyperbolic or elliptic. As a particular case, if the given pairs, being real, be themselves harmonic, the pair harmonic with respect to these will be imaginary.

Ex. 1. Discuss this question algebraically, and show that of the three segments an odd number must be imaginary.

Ex. 2. Show that the double lines of a circular involution are isotropic.

Involutions determined Algebraically.

174. The pairs of points in an involution may be conveniently given by quadratic equations

$$u = ax^2 + 2a'x + a'' = 0,$$
$$v = bx^2 + 2b'x + b'' = 0,$$
$$w = cx^2 + 2c'x + c'' = 0, \text{ etc.},$$

subject to some condition; this can be found from the relations

$$OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC',$$

where O is the centre; let this be the point k; changing the origin to O, the equations become

$$ax^2 + 2ax + a'' = 0, \text{ etc.,}$$

where $$a'' = ak^2 + 2a'k + a'', \text{ etc.,}$$

and $$OA \,.\, OA' = \frac{a''}{a}.$$

Therefore

$$\frac{ak^2 + 2a'k + a''}{a} = \frac{bk^2 + 2b'k + b''}{b} = \frac{ck^2 + 2c'k + c''}{c},$$

hence $$\frac{2a'k + a''}{a} = \frac{2b'k + b''}{b} = \frac{2c'k + c''}{c} = -h,$$

that is, $$ah + 2a'k + a'' = 0,$$
$$bh + 2b'k + b'' = 0,$$
$$ch + 2c'k + c'' = 0.$$

Eliminating h, k, the condition that the three pairs be in involution is

$$\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{vmatrix} = 0.$$

This shows that $$c = la + mb,$$
$$c' = la' + mb',$$
$$c'' = la'' + mb'',$$

and hence that the third quadratic is expressible in the form

$$ax^2 + 2a'x + a'' + \lambda(bx^2 + 2b'x + b'') = 0 \;\ldots\ldots\ldots\ldots(1),$$

that is, $$u + \lambda v = 0;$$

and different values of λ give the quadratics that furnish all pairs of correspondents.

The values of λ that give the double elements make (1) a perfect square; hence they are the roots of

$$(a' + b'\lambda)^2 - (a + b\lambda)(a'' + b''\lambda) = 0.$$

The expression (1) can then be written, after multiplication by $a + b\lambda$, as the square of

$$(a + b\lambda)x + (a' + b'\lambda) = 0,$$

or similarly, as the square of

$$(a' + b'\lambda)x + (a'' + b''\lambda) = 0.$$

Eliminating λ from these, a quadratic in x is obtained whose two roots give the two double elements, viz.,

$$\frac{ax+a'}{bx+b'}=\frac{a'x+a''}{b'x+b''},$$

that is, $(ab'-a'b)x^2+(ab''-a''b)x+(a'b''-a''b')=0.$

The value of λ that reduces the quadratic to a linear equation gives infinity and its correspondent, that is, the centre. Hence for the centre,

$$x=\frac{1}{2}\cdot\frac{a''b-ab''}{ab'-a'b}.$$

175. The two quadratics $u=0$, $v=0$, determine an involution $u+\lambda v=0$ whether their roots be real or imaginary; and this involution certainly contains real pairs. Some real values of λ will give imaginary pairs, but this can happen only if the involution be hyperbolic. For let the centre be origin, then

$$xx'=k,$$

therefore the quadratics are reducible to the form

$$x^2+2a'x+k=0,$$

$$x^2+2b'x+k=0,$$

and $u+\lambda v=0$ is then

$$x^2+2\frac{a'+\lambda b'}{1+\lambda}x+k=0,$$

that is, $$x^2+2px+k=0.$$

Hence if k be negative, so that the involution is elliptic, a real p, which implies a real λ, gives real points. This can be stated:—

Pairs of imaginaries can occur only in a hyperbolic involution; the imaginaries that occur in an elliptic involution are not pairs, in the special sense of § 50.

Hence the involution determined by the quadratics $u=0$, $v=0$ will be hyperbolic unless both quadratics have real roots; in this case it may be hyperbolic or it may be elliptic.

Common Elements of two Involutions.

176. Two involutions on a common base have necessarily one pair of common elements, determined as the pair harmonic to each of the two pairs of double elements. Reference to § 175 shows that these common elements will certainly be

real unless both the given involutions are hyperbolic, in which case they may be real or they may be imaginary.

Considering this case first, let the two pairs of double elements be F_1, F_2; Φ_1, Φ_2. These are real, by hypothesis; hence the required points are determined as the double points of the involution F_1F_2, $\Phi_1\Phi_2$; they are imaginary or real according as the segments (F), (Φ) do or do not overlap.

Now let the first involution be elliptic, F_1, F_2 are imaginary, and the construction just given is inapplicable.

Let the centres of the two involutions be O, Ω, and let a pair of conjugates in each be D, D'; Δ, Δ'. Let K, K' be the common elements, therefore

$$OK \,.\, OK' = OD \,.\, OD', \quad \text{and} \quad \Omega K \,.\, \Omega K' = \Omega\Delta \,.\, \Omega\Delta'.$$

By hypothesis, O is between D, D'; therefore a circle on DD' as diameter contains O; draw the double ordinate S_1OS_2. Join ΩS_1, and take on this a point S_1' such that

$$\Omega S_1 \,.\, \Omega S_1' = \Omega\Delta \,.\, \Omega\Delta'.$$

The circle $S_1S_2S_1'$ cuts $O\Omega$ in the required points K, K'; these are in every case real, since S_1, S_2 are on opposite sides of the line.

This construction applies whether the second involution is hyperbolic or elliptic. If it be elliptic, drawing the double ordinate $\Sigma_1\Omega\Sigma_2$, the circle $S_1S_2\Sigma_1\Sigma_2$ is the one required in the construction.

Note. This construction, like those in § 171, is open to the theoretical objection that it uses circles in a purely descriptive problem. It is however a simple practical construction. A purely descriptive construction is given in § 195.

Involution determined by a Quadrangle.

177. The simplest purely descriptive construction for correspondents in an involution is afforded by the theorem:—

The three pairs of sides of a complete quadrangle are cut in involution by any transversal.

Let the transversal cut the sides in XX', YY', ZZ' (Fig. 41).

Then $$\{A \,.\, BGDZ\} = \{C \,.\, BGDZ\},$$

and as equal pencils determine equal ranges on any line, the ranges determined on the transversal are equal; therefore

$$\{Y'Z'X'Z\} = \{XZ'YZ\},$$

that is, $$\{X'Y'Z'Z\} = \{XYZZ'\},$$

therefore the cross-ratio of four points is equal to that of their conjugates, which shows that the three segments XX', YY', ZZ' are in involution.

Taking for the transversal the special position EF, the points X, X' come together at E, and the points Y, Y' at F; these are therefore the double points of the involution on EF; they are harmonic with respect to the points in which EF is met by BD, AC, agreeing with what we already know as to the harmonic properties of the figure.

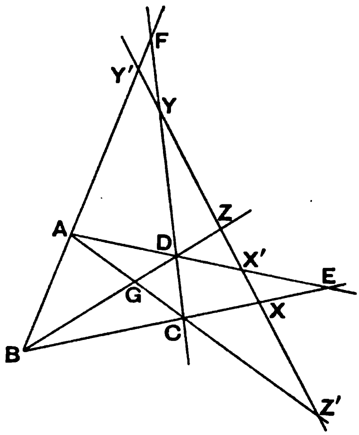

FIG. 41.

Hence to construct Z', the conjugate to Z in the involution determined by XX', YY', it is necessary to construct a complete quadrangle whose sides shall pass through the points. For this the lines through Z, Y, Y' may be taken arbitrarily, but then the points B, D are known, and these must be joined to X, X' (in either order). Thus the points A, C are found, and the line AC passes through Z'. The construction can be differently arranged, and in any special problem the lines of the figure should be utilized as far as possible; the essential thing is to get the three pairs of sides of a complete quadrangle associating the points in pairs as assigned.

178. This theorem shows clearly that the conception of involution is purely descriptive: it depends simply on constructions with collinear points and concurrent lines, and requires no metric determination. The conception of harmonic division is also purely descriptive, as was pointed out in § 45; and harmonic division might be defined accordingly:— If four collinear points be such that a quadrilateral can be described with vertices at two of them, and two diagonals passing through the other two, the points are said to be harmonic.

It is then shown (*e.g.* by means of triangles in perspective)

that this determines uniquely the fourth point, associated with a specified one of the three; and that the segments determined by the two pairs necessarily overlap. (Reye, *Geometrie der Lage.*)

179. That equianharmonic division is also purely descriptive appears from a construction now to be given.

Let 1, 2, 3 be any three collinear points; determine $1'$, $2'$, $3'$ so that $11'$ may be harmonic with respect to 23, etc.; then

(*a*) $11'$ are harmonic with respect to $2'3'$, etc.;

(*b*) $11'$, $22'$, $33'$ are in involution;

(*c*) either triad of points and either double point of this involution form an equianharmonic system.

The analytical proof of this construction illustrates a process of combination that can be used with cross-ratios.

By hypothesis, $(11', 23)$ is harmonic, as is also $(22', 13)$,

therefore $$\{1231'\} = \{2132'\} \quad\text{(i.)}.$$

Continue the range $\{1231'\}$ with $2'$ and then with $3'$, determining the corresponding points on the right.

Let $\{1231'2'\} = \{2132'X\}$; for the determination of X there is any equation made from corresponding cross-ratios on the two sides;

therefore $$\{1232'\} = \{213X\},$$

that is, $$(22', 13) = (1X, 23),$$

hence $(1X, 23)$ is harmonic, and X is $1'$;

therefore $$\{1231'2'\} = \{2132'1'\} \quad\text{(ii.)}.$$

Note. If (i.) be written $\{1231'\} = \{2312'\}$,
which is permissible by harmonic properties, the point X in
$$\{1231'2'\} = \{2312'X\}$$
is determined by $$\{1232'\} = \{231X\},$$
and is therefore $3'$.
Hence $$\{1231'2'\} = \{2312'3'\};$$
and similarly a number of other relations can be found.

Continuing (ii.) with $3'$ on the left,
$$\{1231'2'3'\} = \{2132'1'Y\};$$

therefore $$\{1233'\} = \{213Y\},$$

that is, $$(12, 33') = (21, 3Y),$$

but $(12, 33') = (21, 33')$, being harmonic,

and therefore $$(21, 33') = (21, 3Y),$$

showing that Y is $3'$; hence
$$\{1231'2'3'\} = \{2132'1'3'\} \quad\text{(iii.)}.$$

(*a*) From (iii.), $\{31'2'3'\} = \{32'1'3'\}$,
therefore $(33', 1'2')$ is harmonic; and similarly $(11', 2'3')$, $(22', 1'3')$ are harmonic. Hence starting with $1', 2', 3'$ the construction leads to 1, 2, 3; that is, the two triads of points are symmetrically involved.

(*b*) Since $(11', 23)$ and $(11', 2'3')$ are harmonic,

$$\{1231'\} = \{1'2'3'1\};$$

that is, the cross-ratio of four of the points is equal to that of their conjugates; hence the three pairs of points are in involution.

(*c*) Let x_1, x_2 be the double points of this involution, they are therefore harmonic with respect to every pair $11', 22', 33'$.

Now $12, 1'2'$ are also harmonic with respect to $33'$, therefore $x_1x_2, 12, 1'2'$ are in involution. Hence

$$\{121'x_1\} = \{212'x_2\}$$
$$= \{2'x_2 21\} \quad \text{(iv.)},$$

by permissible interchanges. Therefore

$$\{121'x_1x_2\} = \{2'x_2 21y\},$$

where y is determined by

$$\{11'x_1x_2\} = \{2'21y\},$$

that is, by $\quad (11', x_1x_2) = (2'2, 1y)$.

The left hand is harmonic, hence y must be 3, showing that

$$\{121'x_1x_2\} = \{2'x_2 213\} \quad \text{(v.)}.$$

From (v.), $\quad \{121'x_1x_2 3\} = \{2'x_2 213z\}$,

where $\quad \{121'3\} = \{2'x_2 2z\}$,

that is, $\quad (11', 23) = (2'2, x_2z)$.

The left hand is harmonic, hence z is x_1, and therefore

$$\{121'x_1x_2 3\} = \{2'x_2 213x_1\} \quad \text{(vi.)};$$

and similarly it can be shown that

$$\{212'x_1x_2 3\} = \{1'x_2 123x_1\} \quad \text{(vii.)}.$$

From (vi.), $\quad \{x_1 123\} = \{12'x_2x_1\} \quad$ (viii.);

from (vii.), $\quad \{x_1 123\} = \{2x_2 1'x_1\}$,

that is, $\quad \{2x_1 13\} = \{1'2x_2x_1\} \quad$ (ix.).

Now since x_1, x_2 are the double points of the involution $11', 22'$,

$$\{12'x_2x_1\} = \{1'2x_2x_1\};$$

hence (viii.) and (ix.) give

$$\{x_1 123\} = \{2x_1 13\}$$
$$= \{x_1 231\}$$

(by permissible interchanges), which proves that $\{x_1 123\}$ is equianharmonic, where x_1 is either double point.

And from the involution 11′, 22′, 33′, whose double points are x_1, x_2,

$$\{x_1 123\} = \{x_1 1'2'3'\}.$$

Having found that $\{x_1 123\}$ is equianharmonic, equation (viii.) shows that $\{12'x_2x_1\}$ is also equianharmonic; that is, the two double points with two non-corresponding points from the two triads are equianharmonic.

Hence an equianharmonic range depends on the constructions for harmonic ranges and the double points of an involution; it is therefore purely descriptive.

EXAMPLES.

1. Discuss the contents of this section with reference to the range determined on the line infinity by the three sides of an equilateral triangle and the lines through the vertices perpendicular to the sides.

2. Taking three concurrent lines at equal angles, show that either isotropic line through their intersection completes the equianharmonic pencil.

3. Show that two lines at an angle of 30° and the two isotropic lines through their intersection form an equianharmonic pencil.

4. If the double lines of a pencil in involution be at right angles, they are the bisectors of the angles determined by any pair of conjugates.

5. Show that every pencil in involution contains one pair of orthogonal rays.

Desargues' Theorem.

180. The theorem of § 177 on the involution properties of a quadrangle is a particular case of Desargues' Theorem, which was originally stated with reference to a conic and two pairs of sides of an inscribed quadrangle, but in its general form relates to a pencil of conics. This theorem is:—

Any transversal is cut in involution by a pencil of conics; and reciprocally, *Any point is subtended* (p. 94, footnote) *in involution by a range of conics.*

Using point coordinates, let the base conics be $u=0$, $v=0$, where the transversal is $z=0$, then any conic of the pencil is

$$u+\lambda v=0,$$

and the pairs of points in which this is met by the transversal are given by

$$ax^2+2hxy+by^2+\lambda(a'x^2+2h'xy+b'y^2)=0,$$

and are therefore by § 174 in involution.

If now the conic be chosen to touch the line, the two intersections coincide; the point of contact is a double point of the involution; hence as seen in § 83 two conics can be drawn to pass through four points and touch one line; and two conics can be drawn to touch four lines and pass through one point.

Since the pencil contains three line-pairs, the involution properties of a complete quadrangle are included under Desargues' theorem; and using two of the line-pairs and one proper conic, the original form of the theorem is obtained.

Let the transversal meet the conic in XX', and the line-pairs AB, CD; AC, BD; in PP', QQ'. The fact that the three pairs PP', QQ', XX' are in involution makes the position of X' depend on X, PP', QQ'; that is, on A, B, C, D, X. Thus Desargues' theorem agrees with Pascal's theorem and Chasles' theorem in expressing the dependence of any sixth point of a conic on the determining five points; the three theorems express the same truth under different aspects, and any one can be deduced from any other.

Ex. 1. Deduce the process for finding the centre of an involution (§ 171) from Desargues' theorem.

Ex. 2. Examine the construction for the common elements of two involutions (§ 176) with the help of Desargues' theorem.

Ex. 3. Two conics are determined by five points each, of which three are common. Determine the fourth common point.

General Idea of Involution.

181. Though a formal proof of Desargues' theorem has just been given, yet this is not really necessary; for the definition of involution (§ 167) makes the theorem intuitive. Any conic of the pencil determines a pair of points on the line; and since four points on the conic are known, one point of the pair determines the conic, and therefore determines the other point; hence the elements of the one-dimensional space are associated in pairs, which is the one characteristic of an involution.

The idea here involved has been generalized in various directions. Instead of groups of two points each, we may imagine the points of the line grouped by n's; in the case hitherto considered, two pairs being given by $u=0$, $v=0$, any pair is given by $u+\lambda v=0$; in this generalization, any two groups being given by $u=0$, $v=0$, the involution consists of all groups $u+\lambda v=0$. Since only one parameter, λ, is involved, a single point determines the group. Such a system as this, a singly infinite system of groups of n elements

linearly determined by two groups of the system is called an involution of degree n. For example, a pencil of cubics, $u_3+\lambda v_3=0$, determines on any transversal an involution of degree 3. The involution of pairs of points is therefore of the second degree.

Again, the expressions u, v may be of degree n in any number of variables; that is, instead of representing a group of n points, $u=0$ may represent a curve or a surface of order n; or it may be homogeneous in more than four variables. Thus the pencil of conics $u_2+\lambda v_2=0$ may be regarded as a system of conics in involution.

Just as in the simplest involution there are double elements, so in an involution of degree n there will be certain groups in which some of the elements fall together; and just as in an involution of conics there are three line-pairs, that is, three conics that have double points, so in an involution of curves of higher order there will be certain curves that have double points.

Note. These extensions of the theory of involution can be found in Fiedler, *Die Darstellende Geometrie*, t. iii., pp. 218, 256, 261, etc., and in Clebsch, *Vorlesungen über Geometrie*, t. i., pp. 203, 207-210; where other references will be found.

Another extension depends on an increase in the number of base-expressions u, v, etc. (Cayley, On the Theory of Involution, *Trans. Camb. Phil. Soc.*, 1866; No. 348 in *Collected Papers*, vol. v.), the general element in the involution is then $\lambda u+\mu v+\nu w+\text{etc.}$

182. There is no very convenient distinctive notation in use for involution; occasionally (u, v) may be advantageously used for the involution $u+\lambda v$. If the involution be determined by pairs of points, we may use (AA', BB', CC'); and the double points being F_1, F_2, this can be indicated by a natural extension of the symbolism, $(AA', \ldots, F_1^2, F_2^2)$; or we may denote the whole involution by (F_1^2, F_2^2). It may occasionally be found convenient to write $F=(AA', BB')^{\frac{1}{2}}$, as a symbolic expression of the fact that F is a double element.

Involution Properties of Conics.

183. The intimate association of the theory of involution (of the 2nd degree) with the conic, implied in its definition, leads naturally to a series of theorems expressing this association. For instance, pairs of conjugate points on a line are in involution; this comes from the definition; the known symmetric relation of conjugates proves the (1, 1) correspondence on which involution depends; any point on a conic has itself for one of its conjugates, hence the double

points of the involution are the points in which the transversal meets the conic. Or the theorem can be proved from the fact that conjugate points are harmonic with respect to the two points in which the line joining them cuts the conic. Similarly pairs of conjugate lines through a fixed point form a pencil in involution, and the double lines are the tangents from the point to the conic.

As a particular case of this, conjugate diameters form a pencil in involution, of which the asymptotes are the double lines; these are real or imaginary, that is, the involution is hyperbolic or elliptic, according as the conic is a hyperbola or an ellipse. The involution is circular if the conic be a circle, in which case every pair of conjugate diameters is harmonic with respect to the circular points, that is, every pair of conjugate diameters is at right angles (see § 172).

EXAMPLES.

1. From the fact that two involutions with a common base have one pair of common elements, prove that every central conic has one pair of conjugate diameters at right angles, and that these are necessarily real.

2. Given two pairs of conjugate diameters (in position, not in magnitude), construct the axes and the asymptotes, distinguishing the two cases that occur.

3. Apply Desargues' theorem to show that every conic through the intersections of two rectangular hyperbolas is a rectangular hyperbola.

4. If a pencil contain a circle, the axes of the two parabolas it contains are at right angles; and the axes of all conics of the pencil are in two fixed directions. (Steiner.)

5. If a circle intersect a conic, the pairs of chords of intersection make equal angles with an axis.

184. To obtain the focal properties of conics, consider the two pairs of tangents from ω, ω', and use Desargues' theorem in the reciprocal form. Two conics with the given tangents can be drawn through any point P; their tangents at P are the double lines of the involution formed by the tangents from P to the conics of the range. One conic (degenerate) of the range is the point-pair ω, ω'; hence the tangents to the two conics through P are harmonic with regard to the isotropic lines through P, that is, they are at right angles. Hence confocal conics are orthogonal (compare § 134).

Again, another degenerate conic of the range is the point-

pair F, F' (the real foci); the two perpendicular tangents at P are therefore harmonic with respect to PF, PF', that is, they are the bisectors of the angle FPF'; hence the tangent at any point is equally inclined to the focal distances.

Taking a point T not on the conic considered, let TP, TP' be tangents; the lines joining T to $\omega\omega'$, FF', PP' are in involution; therefore

(i.) the double lines of this involution are at right angles, and consequently

(ii.) they bisect the angle FTF' and also the angle PTP', therefore
$$\angle FTP = \angle F'TP',$$
hence the two tangents that can be drawn from any point to a conic are equally inclined to the focal distances of the point.

185. If any problem depend on the determination of a single point or a single line, we expect to obtain the solution by a linear construction. Not so if the point to be determined be one of a pair; for this we must expect to need a conic. A pair of points, real or imaginary, on a known line, can always serve as the double points of an involution determined by two real pairs, and this is frequently the most convenient way of dealing with a pair that may possibly be imaginary; or again the two points may occur as the double points of two homographic ranges. A construction of this nature, that is, a construction for a pair of elements, is a construction of the second degree.

Ex. Find the points in which a given line meets the conic determined by the five points A, B, C, D, E.

Let one of the required points be P, then by Chasles' theorem the pencils
$$\{A \,.\, CDEP\}, \quad \{B \,.\, CDEP\}$$
are equal. Let AC, AD, AE meet the given line in 1, 2, 3, and let BC, BD, BE meet it in $1'$, $2'$, $3'$; P is either double point of the homographic ranges
$$\{1\ 2\ 3 \ldots\}, \quad \{1'\ 2'\ 3' \ldots\}.$$

Ex. 1. Apply Desargues' theorem to this problem.

Ex. 2. Two conics are determined by five points, of which two are common. Find the line joining the other two common points by a linear construction, and the points themselves by a quadratic construction.

186. A number of relations among segments in involution can be found; these all follow from the two properties, (i.) the cross-ratio of any four points is equal to that of their correspondents; (ii.) the double points are harmonic with respect to every pair of conjugates.

Similarly relations can be found for the segments determined by homographic ranges.

For these reference should be made to Chasles, *Géométrie Supérieure*, Chapters IX. to XIII.; or to Russell, *Pure Geometry*, Chapters X. and XVII. In Ch. XXV. of the latter work there will be found a number of interesting constructions of the second degree.

EXAMPLES.

1. Construct the polar of a point P with regard to the conic determined by A, B, C, D, E.

2. Given a pole and polar, and three points on a conic, construct the conic.

3. Given two poles and polars, and one point on the conic, construct the conic.

4. Given two pairs of conjugates, AA', BB', show that the two points determined by the intersections of the cross-joins are conjugates. (Hesse.)

5. Show that the conics passing through three points and having one assigned pair of conjugates form a pencil; and find a linear construction for the fourth common point.

6. Given four points on a conic, and one pair of conjugates, determine the points in which the conic cuts the join of the conjugates; hence construct the conic.

7. Consider a triangle, and a line-pair; on any side of the triangle there are now two point-pairs, viz., the two vertices, and the intersections with the line-pair; let the double points of the involution on BC be A_1, A_2; etc. Show that the six points $A_1, A_2, B_1, B_2, C_1, C_2$ lie by threes on four straight lines.

8. Apply Desargues' theorem to a conic and a pair of tangents, showing that one double point of the involution determined on any transversal is on the chord of contact of the tangents.

9. Three points and two tangents are given for the determination of a conic; show that the chord of contact is one of four lines; hence construct the four possible conics.

Apply this construction to the case when two of the given points are the circular points, discriminating carefully as to real and imaginary in the construction.

10. Each vertex of a pentagon is the pole of the opposite side with respect to a conic; show that this statement involves exactly the right number of conditions for determining the conic; and find the construction for the conic.

Note. The following four examples relate to the pencil determined by two conics all of whose intersections are

imaginary. All depend on applications of Desargues' theorem.

11. Draw the conic of the pencil that passes through a given point.

12. Draw the conics of the pencil that touch a given line;
   (*a*) if the line touch one conic, and cut the other in (i.) real, (ii.) imaginary points;
   (*b*) if the line cut one conic in real, one in imaginary points;
   (*c*) if the line cut both conics in imaginary points.

13. If the two given conics belong to one "nest," determine one conic of the other nest.

14. Hence construct the common self-conjugate triangle for two conics, all of whose common points and common tangents are imaginary.

15. Take a fixed conic u, a fixed line p, and a fixed point O, these having no special position with regard to one another. Let any variable line through O cut the fixed line in O', and the conic in U, U'; let X, X' be the double elements of the involution (OO', UU'); the locus of X, X' is a conic, ϕ, having O, p as pole and polar.

16. Let v be any other conic having O, p as pole and polar; the intersections of v, ϕ lie on two lines through O.

17. Hence show that through any point O two lines can be drawn to be cut harmonically by two conics u, v.

18. Find the envelope of a line cut harmonically by two conics.

19. Find the locus of a point subtended harmonically by two conics.

20. Find the locus of the intersection of perpendicular tangents to a conic.

21. Any line through a vertex of the self-conjugate triangle of a pencil of conics is cut in involution by the pencil.

22. If the harmonic conic determined by the segments PP', QQ' divide RR' harmonically, then the relation of the three segments is symmetrical. (Clifford.)

23. If P, P' be conjugate with regard to the pencil of conics determined by an orthocentric quadrangle, then with P, P' as foci a conic can be inscribed in the harmonic triangle of the quadrangle.

24. Show that through any point three lines can be drawn to be cut in involution by three conics u, v, w, which do not belong to a pencil.

Determine the lines when the three conics have a common self-conjugate triangle.

25. Show that any line cut in involution by three conics u, v, w is cut in involution by every conic of the net

$$\lambda u+\mu v+\nu w=0.$$

26. Hence show that the envelope of a line cut in involution by a net of conics is a curve of the third class.

Systems of Conics.

187. A number of the examples and theorems discussed have related to special systems of conics; for example, among systems that are singly infinite we have considered a pencil and a range; but there are other singly infinite systems that may be considered, systems of conics determined by three points and one line, by two points and two lines, by one point and three lines. The characteristic of the pencil is that the point equation of any member is linearly expressible in terms of any two members; the fundamental idea in the range is reciprocal to this; the line equation of any member is linearly expressible in terms of any two members.

If then we know that the conics considered in any problem form a singly infinite system, we cannot therefore infer that they form a pencil, or a range. But if the conditions imposed be of such a nature that u, v being members of the system, $u+\lambda v$ is also a member, then all members of the system are given by $u+\lambda v$. For if there be any one, ϕ, not included in this, then by the given conditions all conics $\psi+k\phi$ are included in the system, where ψ is *any one* of the system $u+\lambda v$; hence the system is not a one-fold infinity, for it includes $u+\lambda v+k\phi$, that is, it depends on two independent parameters. Hence we see that all conics of the system *are* given by $u+\lambda v$; if therefore u, v be expressed in point coordinates, the system considered is a pencil; and if u, v be expressed in line coordinates, the system is a range. For instance, given four pairs of conjugate points, that is, four conditions, the conics are singly infinite in number. Let u, v be any two satisfying these conditions; Desargues' theorem shows that all conics $u+\lambda v$ belong to the system; hence all conics of the system are included in $u+\lambda v$, and the conics form a pencil.

Ex. Show that given four pairs of conjugate lines, the conics form a range.

188. A few words may now fitly be said as to the simplest doubly infinite systems of conics, viz., those in which any member is expressible in the form

$$u+\lambda v+\mu w=0.$$

If u, v, w represent expressions in point coordinates, the system is a *net*; the corresponding system with reference to line coordinates has no distinctive name in English, but is sometimes called a tangential net; the name *web* has been suggested.

Note. In French, pencil and range are *Faisceau* and *Système* (this last not exclusively in the sense of range, but generally introduced with an explanation of the meaning intended); net is *Réseau*, and the reciprocal configuration is *Réseau tangentiel.*

In German, pencil of lines and range of points are *Strahlenbüschel* and *Punktreihe*; pencil of conics and range of conics are *Kegelschnittbüschel* and *Kegelschnittschaar;* conics passing through 1, 2, 3 fixed points and touching 3, 2, 1 fixed lines form a *gemischte Kegelschnittschaar*, or simply a *Schaar* (Steiner); net and the reciprocal configuration (web) are *Netz* and *Gewebe.* But usage varies considerably, and in all the word system is frequently used, the precise meaning being stated at the time.

Speaking now only of the net, certain properties are at once evident.

(i.) *All conics that pass through a point form a pencil.*

For if the point be x_1, y_1, z_1, the parameters λ, μ must satisfy

$$u_1+\lambda v_1+\mu w_1=0;$$

eliminating μ, the equation of any conic of the net through x_1, y_1, z_1 is

$$\phi+\lambda\psi=0,$$

where ϕ, ψ are written for

$$uw_1-wu_1, \quad vw_1-wv_1.$$

Hence two points determine a conic uniquely; the points being 1, 2, the conic is

$$\begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix}=0.$$

(ii.) *Any two pencils of the net have a common conic.*

For the pencil determined by A contains one conic that passes through A'; and the pencil determined by A' contains one conic through A; and since two points determine a conic of the net absolutely and uniquely, these two conics must be the same. Moreover, if the A-pencil pass through B, C, D, and the A'-pencil pass through B', C', D', the conic AA' passes through all these points. Hence by means of the net of

conics the points of the plane are grouped in fours, and any two groups lie on a conic.

(iii.) From this, *any conic of the net can be constructed when three conics not passing through a point are known.*

For let the three groups of intersections be

$$A_1,\ B_1,\ C_1,\ D_1;\quad A_2,\ B_2,\ C_2,\ D_2;\quad A_3,\ B_3,\ C_3,\ D_3.$$

Take any point P, the three associated points Q, R, S are given by the conics

$$A_1B_1C_1D_1P,\quad A_2B_2C_2D_2P;$$

and any conic through P, Q, R, S belongs to the net. Hence the statement that two points determine a conic of the net should be:—Two points not belonging to a group determine a conic uniquely.

If it be known that the conics in a doubly infinite system are grouped so that all through any point form a pencil, then the system is a net. For a conic of the system is determined uniquely by two points; now the general equation of a conic of the doubly infinite system contains only two parameters; and since two relations connecting these determine them uniquely, they enter separately and in the first degree. Hence the general conic of the system is

$$u+\lambda v+\mu w=0,$$

and the system is a net.

189. In Ex. 10 after § 82 it is stated that the locus of pairs of points that are conjugate with regard to three conics is a curve of order 3. Now points that are conjugate with regard to three conics are conjugate with regard to their net. For let the transversal be $z=0$, and let the conics be ϕ, ψ, χ, the pairs of points in which the join of the conjugates meets them are expressed by three quadratics, $u=0$, $v=0$, $w=0$, obtained by writing $z=0$ in $\phi=0$, $\psi=0$, $\chi=0$. Since these are by hypothesis in involution,

$$w=au+bv.$$

Taking any other conic of the net,

$$\lambda\phi+\mu\psi+\nu\chi=0,$$

and writing $z=0$, the pair of points is given by

$$\lambda u+\mu v+\nu w=0,$$

that is, by
$$lu+mv=0.$$

Hence this pair is in the involution (u, v), and therefore the given conjugates are conjugate with regard to the conic

$$\lambda\phi+\mu\psi+\nu\chi=0,$$

that is, with regard to any conic of the net.

Hence the locus of the points that are conjugate with regard to a net is a certain order-cubic.

Again, in Ex. 26, after § 186, it is stated that the envelope of a line cut in involution by a net of conics, that is, of a line joining a pair of conjugates, is a class-cubic. Thus associated with the net there is an order-cubic and also a class-cubic; and the whole theory of nets of conics is most satisfactorily studied in connection with curves of the third order or class; not that these higher curves are required for the proofs, but because the results obtained with regard to the net can by their means be more clearly stated and realized.

The theory of cubics as depending on nets of conics is systematically developed geometrically by Schröter, in his *Theorie der Ebenen Kurven dritter Ordnung*; it is algebraically treated by Clebsch, *Vorlesungen*, t. i., pp. 519-527. A detailed account of singly infinite systems of conics, and some discussion of doubly infinite systems, is to be found in Steiner's *Synthetische Geometrie*, t. ii. (Schröter); and the paper "On Some Geometrical Constructions" by H. J. S. Smith (*Proc. Lond. Math. Soc.*, vol. ii., pp. 85-100; 1868) is devoted to a discussion of "systems of order 1, 2, 3, 4" (*i.e.* pencil, net, etc.). But this paper presupposes some knowledge of the theory of Invariants, as does also the treatment adopted by Clebsch.

190. It has been hitherto assumed that the conics u, v, w have not any common points; they may however have in common, one, two, or three fixed points, but not four, for that would reduce the net to a pencil, since plainly any conic

$$\lambda u+\mu v+\nu w=0$$

goes through all points common to u, v, and w. The various theorems stated still hold, except with regard to the fixed points. All conics of the net through any other point form a pencil; but one, two, or three of the base points of the pencil are now fixed; two points, not the fixed points, determine a conic uniquely, and so on.

Determination of a System of Conics by Pairs of Conjugates.

191. We now consider the determination of a conic by means of five conditions, of which a certain number are supplied by pairs of conjugates. This is in close connection with the theory of pencils and nets, and involves the determination of a pencil by four pairs of conjugates, of a net by three pairs of conjugates; and it leads to de Jonquières' solution of Chasles' problem:—*To determine a conic that shall cut five given segments harmonically.**

* *Terquem et Gerono, Annales de Mathématiques*, t. xiv., p. 435 ; 1855.

The proofs of some of the constructions are omitted, as they can easily be supplied from the principles already discussed.

I. *Given four points and one pair of conjugates.*

The pencil determined by the four points cuts the join of the conjugates in an involution; the given conjugates themselves determine on this line a second involution, of which they are the double elements; the common elements of these two involutions are points on the required conic, and with the given four points the determination is complete.

II. *Given three points and two pairs of conjugates.*

Let the given points be P, Q, R, and the conjugates AA', BB'. Describe conics through P, Q, R to touch AA' at A and A' respectively; let S be the fourth intersection of these conics. Any conic of the pencil $PQRS$ has AA' for conjugates, and one conic of the pencil has BB' for conjugates. Hence the required conic is known.

It is better to determine S by a linear construction. Let QR meet AA' in P_1, and let P_2 be conjugate to P_1 with respect to AA'; similarly determine Q_2, R_2. Then QR, PP_2 is one conic of the pencil; therefore PP_2 passes through S, as do also QQ_2, RR_2.

It is here shown that three points and one pair of conjugates determine a pencil.

III. *Given two points and three pairs of conjugates.*

We first show that two points and two pairs of conjugates determine a pencil. Let the points be P, Q, and let the conjugates AA', BB', lie on lines a, b. Let a, b meet in Σ; determine S_1, S_2 on a, b, conjugate to Σ with regard to AA', BB'. Then the conic $PQ\Sigma S_1S_2$ satisfies the given conditions. Let PQ cut a, b in L, M; determine L', M' the conjugates to L, M with regard to AA', BB'; then the line-pair PQ, $L'M'$ satisfies the given conditions. Let $L'M'$ meet the conic $PQ\Sigma S_1S_2$ in X, Y, then any conic through P, Q, X, Y satisfies the given conditions, and all conics are included in this pencil.

The one conic of this pencil that has the third pair of conjugates CC' is the conic determined by the five conditions.

Ex. 1. Show how to construct a conic through the intersections of two given conics, and having one pair of conjugates, in the case when (1) two, (2) four of the intersections of the given conics are imaginary, allowing for the possibility of the line joining the given conjugates meeting one or both of the given conics in imaginary points.

Ex. 2. Describe a circle, when three pairs of conjugates are given.

IV. *Given one point, and four pairs of conjugates.*

That the four pairs of conjugates determine a pencil was shown in § 187; but no construction was given for the

pencil. Let the pairs of conjugates be AA', BB', CC', DD'. Consider the lines AB, $A'B'$, intersecting in X. Any line-pair that is harmonic with respect to these has AA', BB' for conjugates; one such line-pair can be constructed that has CC' for conjugates. Similarly one line-pair can be constructed, with its vertex at X' (the intersection of AB', $A'B$), that has AA', BB', CC' for conjugates; any conic of the pencil determined by these two line-pairs has AA', BB', CC', for conjugates; and one conic of the pencil has also DD' for conjugates. Thus one conic is found having the assigned four pairs of conjugates. Now grouping the pairs differently, another conic is found; and the pencil determined by these two conics has the assigned four pairs of conjugates; it is the required pencil.

One conic of this pencil passes through any given point; hence a conic through one point and having four pairs of conjugates is determined; and one conic of this pencil has a given fifth pair of conjugates; hence

V. *Given five pairs of conjugates*, the conic is determined.

The construction just given can be applied to case III. The three pairs of conjugates can be grouped in three ways: hence three pencils of conics are found having these three pairs of conjugates. Now one point and three pairs of conjugates determine a pencil; hence (§ 188) three pairs of conjugates determine a net; hence the three pencils just found, being three pencils of a net, are not independent, but this does not interfere with the construction. We have to find the conic of this net that passes through the two given points, P, Q. The point P determines one conic of the first pencil and one of the second; these two determine the pencil through P; and the required conic is the one of this pencil that passes through Q.

Hence as concerns the determination of a conic, a pencil, or a net, a pair of conjugates is a single linear condition, equivalent in its effect to a given point.

Note. This is at once apparent algebraically; the condition that two points be conjugate with respect to the general conic imposes a linear condition on the coefficients a, b, etc.

EXAMPLES.

1. Apply the linear construction for the fourth base-point of the pencil determined by three points and one pair of conjugate points to the case of rectangular hyperbolas through three fixed points.

2. Show that the locus of the pole of a fixed line with

respect to a pencil of conics is a conic through five known points, of which three are fixed, and two lie on the line.

3. Apply the result reciprocal to that obtained in Ex. 2 to show that four tangents and one pair of conjugate points determine a conic as one of two. Construct these conics.

4. Construct the conics determined by

(i.) one tangent and four pairs of conjugate points;

(ii.) three points, one tangent, and one pair of conjugate points;

(iii.) two points, one tangent, and two pairs of conjugate points;

(iv.) one point, one tangent, and three pairs of conjugate points.

Homographic Correspondence on Curves.

192. The general definition of homography (§ 160) simply requires the comparison of two one-dimensional spaces; there is to be a (1, 1) correspondence between the elements. Each element being indicated rationally by means of a single parameter μ, μ', the correspondence is expressed by a bilinear relation

$$a\mu\mu'+\beta\mu+\gamma\mu'+\delta=0\dots\dots\dots\dots\dots\dots(1),$$

and the cross-ratio being estimated by means of the values of μ, it follows that the cross-ratio of any four elements is equal to that of their correspondents. This conception is directly applicable to any two unicursal one-dimensional spaces. Thus for example the cubic

$$x^3+y^3-3xyz=0$$

was shown in § 144 to be unicursal; the coordinates of any point are

$$x:y:z=3\mu^2:3\mu:\mu^3+1.$$

Again, the conic $\xi^2=\eta\zeta$ is unicursal, the coordinates of any tangent are

$$\xi:\eta:\zeta=\mu':1:\mu'^2.$$

A (1, 1) correspondence can be instituted between these two one-dimensional aggregates of elements by means of a relation of the form (1). It should be noticed that the bilinear relation between the two parameters does not imply a bilinear relation between the two sets of coordinates, though there may happen to be such a relation. For instance, in the example just given,

$$\mu=\frac{x}{y},\quad \mu'=\frac{\xi}{\eta},$$

therefore (1) becomes

$$\alpha x\xi+\beta x\eta+\gamma y\xi+\delta y\eta=0\,;$$

but the relation satisfied by

$$y:z \quad\text{and}\quad \eta:\zeta$$

is of a different form.

193. The two homographic systems may be in the same one-dimensional space; for example, we may have homographic ranges on a conic, just as we have homographic ranges on a line. And instituting a (1, 1) correspondence between the points of a conic, we have an involution on the conic.

It was shown in § 142 that the coordinates of a point on a conic can be expressed in terms of a single parameter in a doubly infinite number of ways. But whatever system of expression may be adopted, the cross-ratio of the parameters of four points is the same, and is the cross-ratio of the pencil determined in the conic. For let the expressions be

$$x:y:z=a\mu^2+a'\mu+a'':b\mu^2+b'\mu+b'':c\mu^2+c'\mu+c''\,;$$

knowing that the pencil determined at any fifth point of the conic is the same whatever point be taken, we can take the pencil subtended at $\mu=0$. The ray of this that passes through a point μ is

$$\begin{vmatrix} x & y & z \\ a\mu^2+a'\mu+a'' & b\mu^2+b'\mu+b'' & c\mu^2+c'\mu+c'' \\ a'' & b'' & c'' \end{vmatrix}=0\,;$$

subtracting the last row from the second, and then dividing the second row by μ, this becomes

$$\begin{vmatrix} x & y & z \\ a\mu+a' & b\mu+b' & c\mu+c' \\ a'' & b'' & c'' \end{vmatrix}=0.$$

Hence any ray of the pencil is $u+\mu v$, where

$$u=x(b'c''-b''c')+y(c'a''-c''a')+z(a'b''-a''b'),$$
$$v=x(bc''-b''c)+y(ca''-c''a)+z(ab''-a''b),$$

and the pencil is therefore $(\mu_1\mu_2,\ \mu_3\mu_4)$, that is, by § 41, a pencil of cross-ratio

$$\frac{\mu_1-\mu_3}{\mu_3-\mu_2}:\frac{\mu_1-\mu_4}{\mu_4-\mu_2}.$$

In dealing with systems of points on a conic, we are therefore at liberty to adopt the most convenient system of parametric expression. For instance, taking two tangents and

their chord of contact for the lines x, y, and z, the equation of the conic is of the form

$$xy = z^2,$$

and any point on it is $1, \mu^2, \mu$. This form is discussed in Salmon's *Conic Sections*, §§ 270–277. It is there shown, among other things, that points $\pm\mu$ are collinear with C. Now lines through C evidently associate the points of the conic in pairs, they therefore determine an involution; and since the parameters of the points forming a pair are now $\mu, -\mu$, the relation is

$$\mu' = -\mu, \text{ that is, } \mu + \mu' = 0,$$

a special form of the symmetrical bilinear relation that expresses involution.

194. The fact that concurrent lines determine an involution on a conic is here shown to follow at once from the definition of involution; and similarly there follows directly the converse theorem:—The joins of correspondents in an involution on a conic are concurrent. For just as in the case of an involution on a straight line, two pairs determine the arrangement; let their joins AA', BB' meet in O. Then lines through O determine an involution in which AA', BB' are correspondents, that is, they determine the involution in question. Correspondents PP' coincide if the line OP be a tangent; hence in the involution there are two double elements F_1, F_2, these being points of contact of tangents from O. The point is called the pole of the involution, and F_1F_2, the polar of O, is the axis. The known properties of poles and polars show that AB, $A'B'$, as also AB', $A'B$, meet on the axis. Moreover since any chord PP' passes through O, it is conjugate to F_1F_2; therefore (Ex. 7, § 89) the points (F_1F_2, PP') are harmonic; that is to say, the double points of the involution are harmonic with respect to any pair of correspondents.

The common elements of two involutions on a conic are at once constructed, for they are determined by the line that joins the two poles.

195. The theory of involution on a conic gives purely descriptive constructions for the double elements of an involution on a line, and for the common elements of two involutions on a line. It has been shown that the cross-ratio of four points on a conic is determined by the cross-ratio of the pencil subtended at any point of the conic; but this is also the cross-ratio of the range on any transversal. Hence the involution on a line can be projected on to any conic

by means of lines joining its elements to a fixed point on the conic. The pole of the involution on the conic is determined by two pairs; by means of the pole the double points are known; and projecting these on to the original line, the double elements of the original involution are found. Similarly two involutions on a line can be projected on to a conic, and their common elements are therefore found. It will be seen at once that if one involution be elliptic, the pole of the corresponding involution on the conic is inside the conic; hence the line joining the two poles certainly cuts the conic in real points, that is, the common elements of the two involutions are real (§ 176).

196. Since the cross-ratio of four points on a conic is the cross-ratio of the pencil subtended at any fifth point, in comparing two ranges, the pencils formed may have different vertices; the relation of homographic ranges may therefore be stated:—Homographic ranges on a conic subtend homographic pencils at any two points of the conic. This is the foundation of the geometrical treatment of the topic; it gives the theorem:—

The intersections of cross-joins of pairs of correspondents are collinear.

Let pairs of correspondents be AA', BB', etc.; then

$$\{A \,.\, A'B'C'\ldots\} = \{A' \,.\, ABC\ldots\}.$$

These equal pencils have a common ray, AA', hence intersections of other correspondents are collinear; that is, the intersections of $(AB', A'B)$, $(AC', A'C)$, etc., are collinear. Thus by means of pencils at A, A', a certain line a is determined; similarly lines b, c are determined; these are to be shown the same. The hexagon $AB'CA'BC'$ shows that

$$(AB', A'B), \quad (BC', B'C), \quad (CA', C'A)$$

are collinear; and these three points in twos determine a, b, c; hence all these lines are the same. The line on which these intersections lie is called the homographic axis.

Most of the properties of homographic systems on a conic follow at once from the bilinear relation beween the parameters, exactly as in the case of homographic ranges on a line. Three pairs of correspondents determine the system; there are two common points; etc.

Note. The chapters on homographic ranges on a conic, and on involution on a conic, in Russell's *Pure Geometry* (XVI. and XX.) may be referred to for examples on these theories.

For cross-ratio properties of conics, Chapter XVI. in Salmon's *Conic Sections* should be read.

CHAPTER X.

PROJECTION AND LINEAR TRANSFORMATION.

Effect of Projection.

197. We now return to the theory of projection, of which the fundamental idea is explained in §§ 148-150. A plane figure is projected from one plane on to another by means of lines through a fixed point V, the centre of projection; the two figures agree as to descriptive and anharmonic properties. Since any line and its projection meet on the line of intersection of the two planes, this line is as important in the diagram as the centre of projection; call it XYZ, or v. If V, v, and A', the projection of any one point A, have been marked in, the projection of any other point B can be at once inserted. For let AB meet v in X (Fig. 36); then XA' passes through B'; and as VB also passes through B', this point is found.

198. A line parallel to v in the original plane projects into a line parallel to v; for a plane containing a parallel to v is itself parallel to v, and is cut by any plane containing v in a line parallel to v; hence a system of lines parallel to v projects into a system of parallel lines. But any other system of parallels projects into a system of concurrent lines. To prove this, note in the first place that the relation of two figures is symmetrical, that is to say either may be regarded as a projection of the other. Let lines p_1, p_2 meet in P (Fig. 42); project on to a plane parallel to PV; let p_1, p_2 meet the line of intersection in X, Y. By elementary solid geometry, planes VPX, VPY are cut by the plane parallel to VP in parallel lines p_1', p_2'. Moreover the projection of P is the intersection of VP with the plane parallel to VP. Hence the concurrent lines p_1, p_2 are projected into parallel lines, their intersection being projected to infinity. Now let there be a line through P parallel to XY; this is projected by a plane parallel to the plane of projection,

and every point on it is projected to infinity; but no other point in the plane PXY is projected to infinity, for every other point has a finite projection. Hence we see that in a plane, infinity is a straight line. In a diagram the line in each plane that represents infinity in the other plane is of importance.

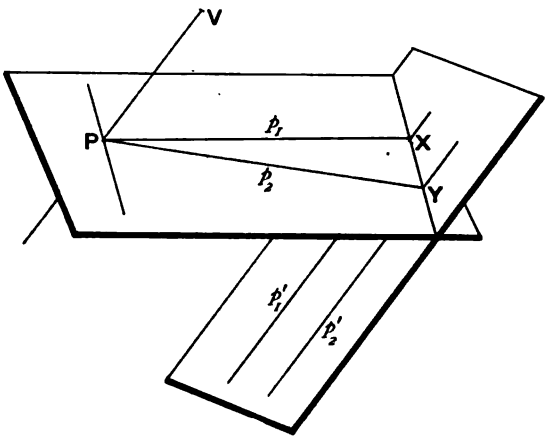

FIG. 42.

From the reciprocal nature of the two figures in Fig. 42, it is seen that parallel lines project into a pencil of lines. In order to project a pencil of lines into parallel lines, the cutting plane must be parallel to VP; and to project a second pencil of lines with vertex Q into parallel lines, the cutting plane must be parallel to VPQ. This leaves us free to choose V arbitrarily, but then the direction of the cutting plane is determined. Hence any line can be projected to infinity from an arbitrarily chosen centre.

For example, a quadrilateral can be projected into a parallelogram; in Fig. 41 let EF be projected to infinity, $ABCD$ becomes a parallelogram, and consequently BD, AC bisect each other in G. Let BD meet EF in H; then the relation can be stated in a projective form, for H being at infinity, (BD, GH) is harmonic. Thus the harmonic properties of a complete quadrilateral are deduced from the properties of a parallelogram.

Possibilities of Projection.

199. We know that the magnitudes of lines and of angles are altered by projection; but we have just seen that we can

to a certain extent determine beforehand what this alteration shall be. We consider therefore to what extent we can in general control this alteration.

It was shown in § 151 that the ratio of two segments on a line can be altered to any desired extent, but that other segments on the line are then determined. Now let there be segments not all on one line; let ABC be on one line, PQR on another; we can project so that $AB:BC$ and $PQ:QR$ may assume any desired values λ and μ. To do this, take D on ABC and S on PQR so that

$$(AC, BD) = -\lambda, \qquad (PR, QS) = -\mu,$$

and project DS to infinity.

Ex. Project so that two non-collinear segments may be bisected at their intersection.

200. The extent to which we can control the alterations in linear and angular magnitudes is assigned in the most generally convenient form by the theorem:—

Any straight line can be projected to infinity, and at the same time any two angles into given angles.

For in any projection let the plane through V parallel to

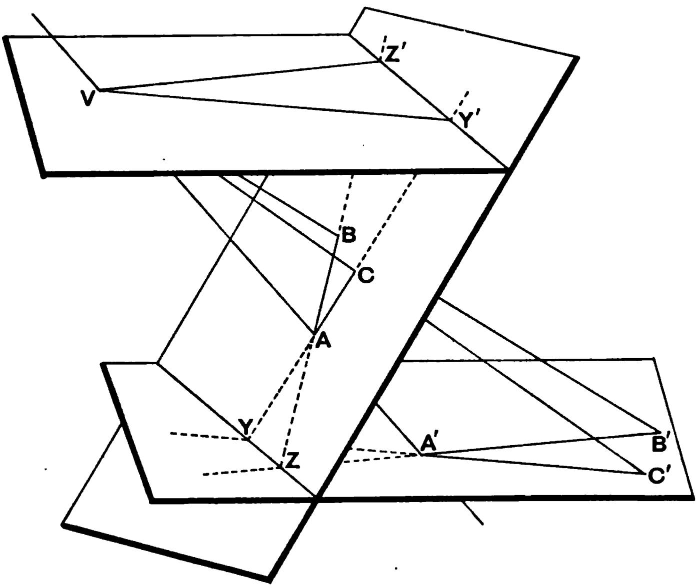

FIG. 43.

the cutting plane meet the original plane in v'. Let the lines AB, AC containing an angle A cut v, v' in ZZ', YY'

(Fig. 43). Then VY', VZ' are parallel to $A'Y$, $A'Z$, and therefore the angle A' is equal to the angle $Y'VZ'$; also v' is projected to infinity. Hence to project so that a particular line v' may go to infinity, while two angles BAC, QPR assume given magnitudes θ, ϕ:—

Let the sides AB, AC cut v' in Z', Y', and let PQ, PR cut v' in R', Q'; take *any* plane through v', in this on $Y'Z'$ describe a segment of a circle containing an angle θ, and on $Q'R'$ describe a segment containing an angle ϕ; let V be an intersection of these. From V as a centre project on to *any* plane parallel to Vv', then the angles assume the required magnitudes, and the assigned line goes to infinity.

As the circles may possibly intersect in imaginary points, it may require an imaginary projection to accomplish the desired transformation.

Ex. 1. Any triangle can be projected so as to be similar to a given triangle, while any assigned line goes to infinity.

Let BC, CA, AB cut the assigned line in X', Y', Z' (Fig. 44). On $Y'Z'$, $Z'X'$ describe segments containing the required angles α, β. These

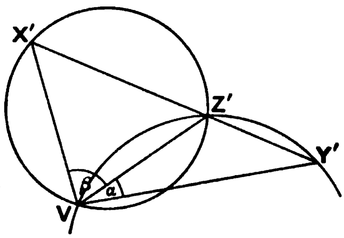

FIG. 44.

segments have one real intersection Z', and therefore they have another real intersection V. Hence the transformation is accomplished by a real projection.

Ex. 2. A quadrilateral can be projected into a square by a real projection.

One diagonal must be projected to infinity, and the angles formed by (1) the remaining diagonals and (2) one pair of sides must be projected into right angles.

Ex. 3. Any conic with any point can be projected into a circle and its centre.

Take any two pairs of conjugate lines through the point. Project so that these may be at right angles, while the polar of the point goes to infinity. The conic remains a conic; the point becomes the pole of infinity, that is, the centre; the pairs of conjugate lines become conjugate diameters; but a conic with two pairs of conjugate diameters at right angles is a circle; hence the projection is accomplished, but it is not necessarily real.

201. This last example gives the important result that *any two points can be projected into the circular points.* For taking the line joining the points, and any conic through the points, the result is attained by projecting so that the conic becomes a circle while the line goes to infinity.

Ex. 1. Show that if the two points be real, the projection is imaginary, and vice versa.

Ex. 2. Show that three angles with different vertices can be projected into right angles.

Comparison of Different Projections.

202. In the proof of the theorem in § 200 there is a certain margin of choice in the construction:—

(i.) take *any* plane through v';

(ii.) project on to *any* plane parallel to Vv'.

Two different determinations under the heading (ii.) give similar figures, the ratio of their linear dimensions being as the distances of the planes from V. It remains to examine the effect of a different determination under the heading (i.). Let the centres be V_1 and V_2; the planes on to which we project are parallel to V_1v' and V_2v', and may be taken to meet the original plane in the same line v. Since the figures in the planes V_1v', V_2v' are exactly the same, one can be brought to coincidence with the other by a rotation about v' as an axis. We have to show that a similar thing is true about the two projections.

Constructing a diagram like Fig. 43 for each projection, and using suffixes for the points in the projections, we have

$$ZA_1 : V_1Z' = ZA : AZ',$$

$$ZA_2 : V_2Z' = ZA : AZ';$$

now $$V_2Z' = V_1Z',$$

therefore $$ZA_2 = ZA_1.$$

Also $$\angle A_1ZY = \angle V_1Z'Y',$$

$$\angle A_2ZY = \angle V_2Z'Y';$$

now $$\angle V_2Z'Y' = \angle V_1Z'Y',$$

therefore $$\angle A_2ZY = \angle A_1ZY;$$

and similarly for other lines and angles in the two projections. Hence the second projection is simply the first with its plane turned through a certain angle.

Thus the apparent choice relates simply to size and to posi-

tion in space. The projection is absolutely determinate as to shape * when the magnitudes of two angles, and the line that is projected to infinity, are given. The plane of projection may make any angle with the original plane without any alteration in the resulting figure.

203. Finally we may suppose the plane on to which we project to revolve so as to come into coincidence with the original plane; then the Vv' plane also coincides with this. The whole figure is now in one plane, with its general properties unaltered; corresponding lines intersect on v, the axis, and corresponding points connect through V, the centre. Thus we are led to the theorem:—

When two figures in a plane are related so that the joins of corresponding points are concurrent, then the intersections of corresponding lines are collinear.

The point and line are Poncelet's centre and axis of homology; thus the theory of homology, or perspective, is rendered self-evident by means of perspective in space.

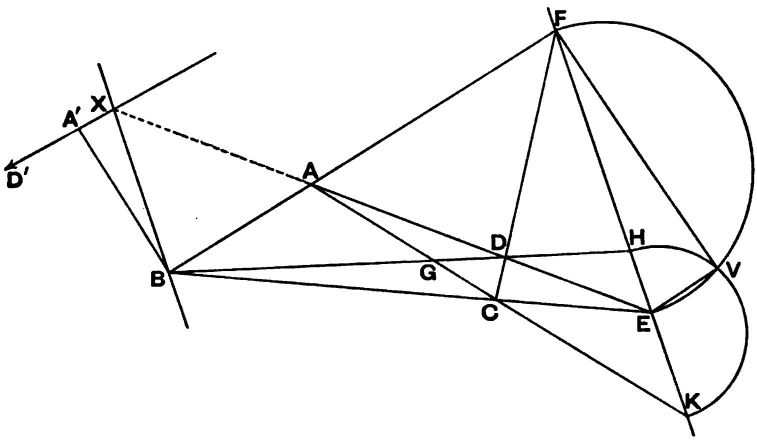

Fig. 45.

As an example of this plane projection, consider the transformation of a quadrilateral into a square.

* The segments of circles used for the determination of V may have their intersections, when these are real, on opposite sides of the line, in which case only one of the intersections satisfies the conditions of the problem unless the segments be semicircles; or on the same side of the line. In this case there are two points V that might be taken as centre of projection; and the projection is one of two. This cannot occur when we project so that two angles may become right angles, which is the usual projection.

Let the opposite sides of the quadrilateral $ABCD$ (Fig. 45) meet in E, F, and let the diagonals, which intersect in G, meet EF in H, K. The line EF must be projected to infinity, while the angles FBE, HGK become right angles. On EF, HK, describe semicircles, let these intersect in V; V is the centre of perspective. For axis of perspective any line parallel to EF is to be taken; take for example the line through B parallel to EF. Let AD cut this in X; then $A'D'$ goes through X, and is parallel to VE; hence $A'D'$ is constructed. Similarly the other lines are constructed, and the resulting figure is a square. In Fig. 45, the projections of the lines AD, BA are marked, their intersection is the point A', which lies on VA, as it should; and completing the figure by means of the projections of BC, CD, the points C', D' will be found to lie on VC, VD.

204. Two distinct projections of a figure, with a common line for the three planes, are projections of one another; for the common line is the axis of perspective. Let the two centres be V_1, V_2, and let the derived centre be V'; these three centres are collinear. For if the two projections of AB be A_1B_1, A_2B_2, then AB, A_1B_1, A_2B_2 meet on the axis of perspective; therefore the triangles AA_1A_2, BB_1B_2 are in perspective; and therefore the points

$$(A_1A_2,\ B_1B_2),\quad (A_2A,\ B_2B),\quad (AA_1,\ BB_1)$$

are collinear; that is, V', V_1, V_2 are collinear.

Alteration in Appearance caused by Projection.

205. Each of the two planes is divided into three compartments by the line of intersection, the line infinity, and the line that is the projection of infinity on the other plane; and these three compartments of one plane project into the three compartments of the other taken in a different order. In Fig. 46, ∞J, JX, $X\infty$ in one plane project into $I'\infty$, ∞X, XI'.

The appearance of the figure is most altered by projection when it cuts either the line infinity or the line that corresponds to infinity in the projection. To insert the projection of any point R accurately, when one pair of lines PX, $P'X$ is drawn, let RM parallel to v cut PX in M; let MV cut $P'X$ in M'; let $M'R'$ parallel to v cut RV in R'; R' is the projection of R. In Fig. 46 two points R, S, off the line PQ, are marked in this way; these have been chosen in about the same positions relative to P, Q respectively; but being in different compartments, the projections present a different

appearance. By inserting a few points in the neighbourhood of J, on the two sides of the line, the effect of projection

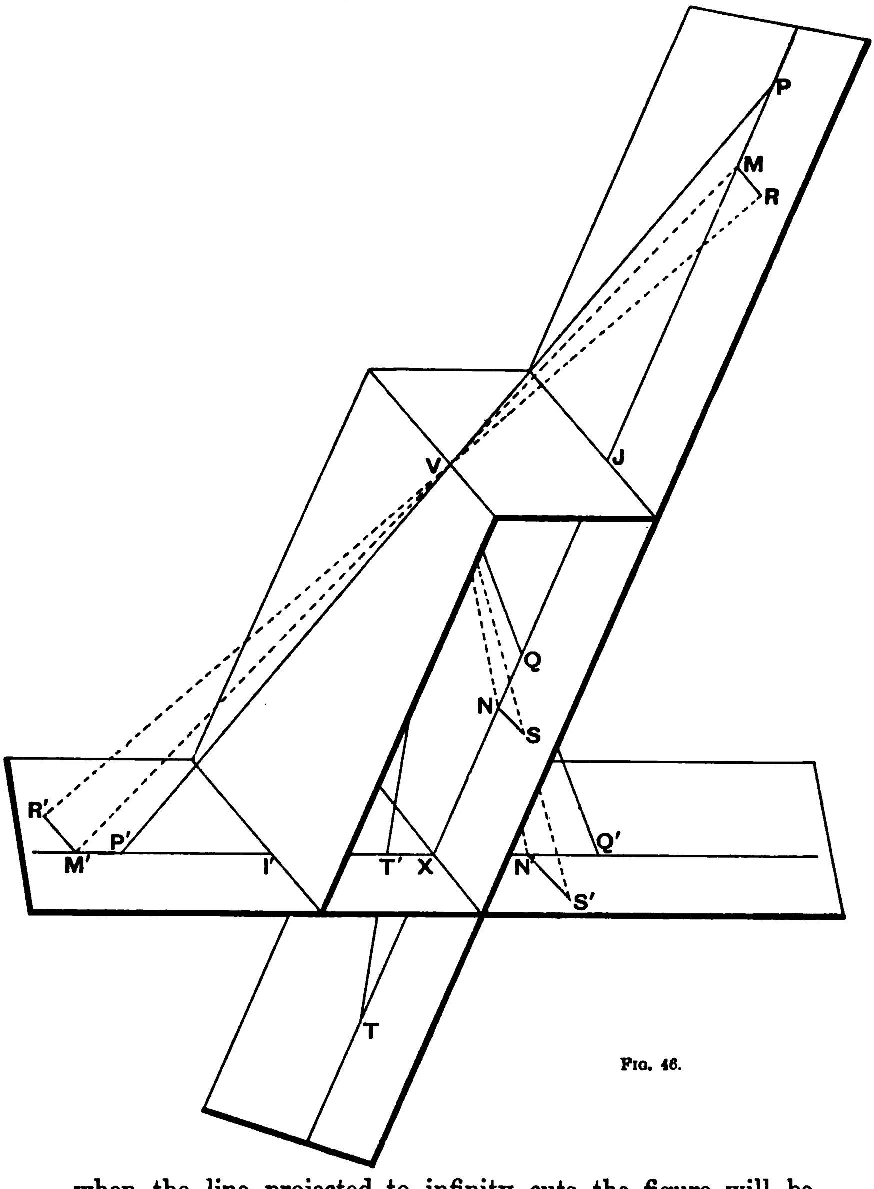

FIG. 46.

when the line projected to infinity cuts the figure will be made apparent. The effect should be noticed for the various

cases that arise. The line through J may

(i.) cut the curve; let PQ be the tangent;

(ii.) touch the curve; let the point of contact be on PQ.

(i.) The point may be

(*a*) an ordinary point, Fig. 47, (i) a, a;
(*b*) an inflexion, (i) b, b;
(*c*) a crunode;
(*d*) a cusp, (i) d, d.

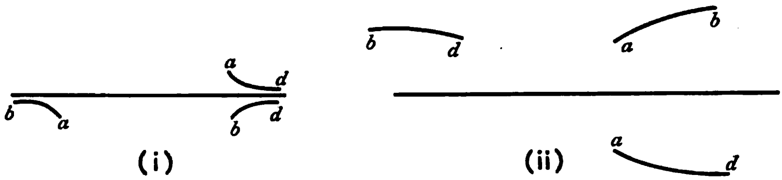

FIG. 47.

(ii.) The line may be

(*a*) an ordinary tangent, Fig. 47, (ii) a, a;
(*b*) an inflexional tangent, (ii) b, b;
(*c*) a tangent at a crunode;
(*d*) a tangent at a cusp, (ii) d, d.

(i.) (*c*) is not represented in the diagram, as it is simply (i.) (*a*) taken twice; and (ii.) (*c*) is a combination of (i.) (*a*) and (ii.) (*a*).

206. Since any conic can be projected into a circle while any desired line goes to infinity, in proving any descriptive theorem for a conic it is sufficient to prove it for a circle. Thus, for example, Pascal's theorem need only be proved for a circle, and the inscribed hexagon can be taken with two pairs of opposite sides parallel; the truth of the theorem is at once evident, for the third pair of sides is necessarily parallel. Again, the constancy of the cross-ratio of a pencil in a conic is deduced from the constancy of the angles in a segment of a circle. The use of projection in proving properties of conics is fully treated in Salmon's *Conic Sections*, Chapter XVII., where, in § 366, there is the formal proof by elementary geometry that any conic can be projected into a circle. It should be noticed that a system of conics with two common points can be projected into a system of circles, and a pencil of conics into coaxal circles; and that a range of conics can be projected into a system of confocals.

Ex. 1. Draw the curve

$$(yz-2x^2)(yz-\tfrac{1}{2}x^2)=xy^2z.$$

Apply § 205 to determine the appearance according as one or other

of the three lines x, y, z is projected to infinity; and verify by drawing the curves whose Cartesian equations are obtained by writing 1 for x, y, z respectively.

Ex. 2. Do the same thing with the curves in § 147, Ex. 1.

Analytical Aspect of Projection.

207. We have now to consider how projection presents itself as a method in analytical geometry.

Let the sides of the fundamental triangle be a, b, c. Suppose we are working with actual perpendiculars, then the present line infinity is

$$a\alpha+b\beta+c\gamma=0.$$

We project so that

$$p\alpha+q\beta+r\gamma=0$$

may go to infinity, and so that the triangle may have sides a', b', c'. Any straight line projects into a straight line, that is,

$$A\alpha+B\beta+C\gamma=0$$

becomes

$$A'\alpha'+B'\beta'+C'\gamma'=0;$$

hence any linear function of α, β, γ becomes a linear function of α', β', γ'; we have therefore a *linear transformation*. It is a *special* linear transformation, for the lines

$$\alpha=0, \quad \beta=0, \quad \gamma=0,$$

are to become

$$\alpha'=0, \quad \beta'=0, \quad \gamma'=0;$$

hence the transformation is of the form

$$\alpha=\lambda\alpha', \;\; \beta=\mu\beta', \;\; \gamma=\nu\gamma'.$$

By this the line

$$p\alpha+q\beta+r\gamma=0$$

is to become

$$a'\alpha'+b'\beta'+c'\gamma'=0;$$

therefore

$$p\lambda : q\mu : r\nu=a' : b' : c',$$

that is,

$$\lambda : \mu : \nu=\frac{a'}{p}:\frac{b'}{q}:\frac{c'}{r};$$

hence the transformation is

$$\alpha=\frac{a'}{p}\alpha', \;\; \beta=\frac{b'}{q}\beta', \;\; \gamma=\frac{c'}{r}\gamma'.$$

Now a', b', c', $p:q:r$ may initially be chosen arbitrarily; consequently $\frac{a'}{p}$, $\frac{b'}{q}$, $\frac{c'}{r}$ can be made to assume any values l, m, n; and the formulæ of transformation are

$$\alpha=l\alpha', \;\; \beta=m\beta', \;\; \gamma=n\gamma'.$$

Hence the use of projection in space shows that if we prove a projective theorem using actual perpendiculars, the same

work proves the theorem when the coordinates are any multiples we please of the perpendiculars. We are therefore independent of the nature of the coordinates in dealing with descriptive and projective metric properties; that is, we may take any equation we please for the line infinity.

208. In the case of plane projection, it is better to refer both figures to the same fundamental triangle. Let the axis be

$$fx+gy+hz=0;$$

the line x' goes through the intersection of this and the line x, hence

$$\left.\begin{aligned} x' &= px+\lambda(fx+gy+hz), \\ y' &= qy+\mu(fx+gy+hz), \\ z' &= rz+\nu(fx+gy+hz), \end{aligned}\right\} \quad \text{(i.)};$$

the multipliers implied in x', y', z' may be chosen so that λ, μ, ν, $=1$, and then the formulæ of transformation become

$$\left.\begin{aligned} x' &= f'x+gy+hz, \\ y' &= fx+g'y+hz, \\ z' &= fx+gy+h'z, \end{aligned}\right\} \quad \text{(ii.)}.$$

The line joining the point A' (*i.e.* $y'=0$, $z'=0$) to the point A is

$$(g-g')y=(h-h')z,$$

and similar equations hold for BB', CC'; hence the centre of projection is given by the equations

$$(f-f')x=(g-g')y=(h-h')z \quad \text{(iii.)}.$$

The interpretation of these formulæ is that if $F(x, y, z)=0$ be any curve, then the equation of the curve obtained by a certain projection is $F(x', y', z')=0$, where x', y', z' are expressed linearly in terms of x, y, z by means of equations (ii.). Hence projection is accomplished by means of linear transformation.

209. As an example, let it be required to project the curve

$$x^3+2xy^2-2x^2-3y^2+x=0 \quad \text{(i.)},$$

so that $x+y-1=0$ may go to infinity, the curve remaining unaltered in the immediate neighbourhood of O.

To do this, we (1) change to homogeneous coordinates, and then (2) change the triangle of reference in part so that the assigned line may be the third side; and finally (3) project this third side to infinity.

By (1) equation (i.) becomes

$$x^3+2xy^2-2x^2z-3y^2z+xz^2=0 \quad \text{(ii.)}.$$

The line $x+y-z=0$ is to be taken as the line z'; hence we have the equations

$$x=\lambda x', \quad y=\mu y', \quad x+y-z=\nu z'.$$

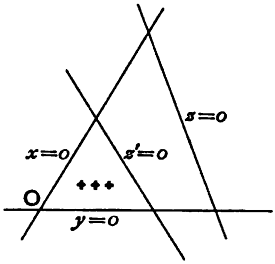

FIG. 48.

As the equation is to be unaltered for small values of x and y, $\lambda=1$, $\mu=1$. Noticing how the line z', that is, $x+y-z=0$, lies, it is seen that in order that a point inside the new triangle may have its three coordinates positive, ν must be negative; for $x+y-z$ is negative at a point inside this triangle. Write $\nu=-p$, and the formulæ of transformation become

$$x=x', \quad y=y', \quad z=x'+y'+pz'.$$

Making these substitutions and dropping the accents, equation (ii.) becomes

$$x^3+2xy^2-2x^2(x+y+pz)-3y^2(x+y+pz)+x(x+y+pz)^2=0,$$

which reduces to

$$-3y^3+(2xy-3y^2)pz+x(pz)^2=0\ldots\ldots\ldots\ldots\text{(iii.)}.$$

This is the result of step (2) in the process; (3) is accomplished by writing for z any convenient constant (see § 30); we write therefore $pz=1$, and the equation of the projection becomes

$$3y^3-2xy+3y^2-x=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv.)}.$$

This can be verified by drawing the curves (i.) and (iv.), and applying § 205, by means of which it will be seen that the change in shape is what it should be for the given projection. The second curve has a node at infinity, the two tangents being the line infinity and a line parallel to the axis of x.

In general, unless the line that is to go to infinity passes through O, it is best to keep x, y unchanged. Let the line to be projected to infinity be

$$ax+by=cz,$$

then the formulæ of transformation are

$$x=x', \quad y=y', \quad -ax-by+cz=pz',$$

from which $\qquad cz=ax'+by'+pz'$;

here the sign of p must be carefully determined, as in the example given above; the numerical value is indifferent, for at the end any convenient constant is written for z', and therefore for pz'. Thus p can be taken numerically unity, but it may be $+1$ or -1.

Ex. Find formulæ for projecting

(i.) $3x+3y+1=0$, (ii.) $x-y+2=0$, (iii.) $x-1=0$,

to infinity; and apply these as follows:—

(i.) to $x(x-y)^2+7x^2-6xy+3x+3y+1=0$;

(ii.) to $x(x-y)^2+7x^2-7xy+3x+3y+1=0$;

(iii.) to $x(x-y)^2+x^2-y^2+y=0$.

Verify the results by drawing the curves, and applying § 205.

General Linear Transformation.

210. Projection has here presented itself as a specialized linear transformation; before considering to what extent the transformation is specialized, we must consider the effect of linear transformation in general. In this we write

$$\left.\begin{aligned} x' &= l_1x+m_1y+n_1z,\\ y' &= l_2x+m_2y+n_2z,\\ z' &= l_3x+m_3y+n_3z, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)},$$

where the coefficients are independent. But these are the formulæ for changing the fundamental triangle from xyz to $x'y'z'$, where $x'=0$ is the line

$$l_1x+m_1y+n_1z=0.$$

For example, let it be required to transform so that the new triangle of reference may be formed by the tangents at A, B, C to the conic

$$F=fyz+gzx+hxy=0.$$

These tangents are $\dfrac{y}{g}+\dfrac{z}{h}=0$, etc., hence the formulæ of transformation can be written

$$\frac{y}{g}+\frac{z}{h}=2\lambda x', \quad \frac{x}{f}+\frac{z}{h}=2\mu y', \quad \frac{x}{f}+\frac{y}{g}=2\nu z';$$

whence $$\frac{x}{f}=-\lambda x'+\mu y'+\nu z', \text{ etc.}$$

Suppose for clearness that f, g, h are all positive, so that the position of the new triangle with regard to the old is that represented in Fig. 49 (a). A point inside the original triangle is also inside the new triangle, and therefore λ, μ, ν are all positive.

Making these substitutions, and arranging, the equation of the conic becomes

$$F'=\lambda^2x'^2+\mu^2y'^2+\nu^2z'^2-2\mu\nu y'z'-2\nu\lambda z'x'-2\lambda\mu x'y'=0,$$

the ordinary form for the equation of a conic inscribed in the triangle $x'y'z'$.

These two equations $F=0$, $F'=0$ have been found as two different equations of one curve, with two different triangles of reference. But dropping the accents from x', y', z', the second equation has a meaning when referred to the original triangle; it represents a conic inscribed in this triangle. Thus the two equations

$$F=0, \quad F'=0,$$

may be connected in two different ways, as represented in Fig. 49 (a) and (b).

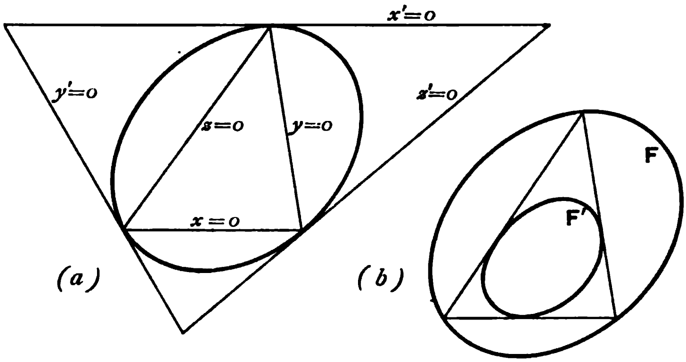

FIG. 49.

211. Adopting the second way of regarding the two equations, we proceed to consider more carefully the correspondence of the two figures, these not being restricted to the special example of the last section.

In the one figure F, we have a point x, y, z: from this, by equations (i.) of § 210, we derive a point x', y', z'; hence point corresponds to point. Moreover, the equations of transformation being linear, straight line corresponds to straight line. Hence all properties of collinearity, concurrence, order, class, etc., that is, all descriptive properties, are the same for the two figures. Further, cross-ratio is unaltered; for if linear expressions u, v become u', v', expressions $u+kv$, $u+lv$ become $u'+kv'$, $u'+lv'$; and the cross-ratio of the configuration, being $k:l$, is unaltered. Hence when from any figure another is derived by linear transformation, the two agree as to all projective properties.

Comparison of Projection and Linear Transformation.

212. The number of constants involved in the general linear transformation being 9, the number of disposable constants is 8, for we are concerned only with ratios. In the formulæ

for projective transformation, § 208, (ii.), the number of coefficients is 6, and there are therefore 5 disposable constants; projection presents itself as a special case of linear transformation.

But the difference is simply one of position. Let a figure Φ be derived from F by linear transformation. Suppose that F by projection becomes F_1, move F_1 about in the plane in the most general way possible, that is, turn F_1 through an arbitrary angle θ, and bring any assigned point of F_1 to an arbitrary point whose (Cartesian) coordinates are a, b. Let F_1 in this new position be called F'. Hence F has become F' by

(1) projection, involving 5 independent constants,

(2) displacement, involving 3 independent constants;

that is to say, F has become F' by a linear transformation involving 8 independent constants; and these can be determined so that the transformation may be the one by which Φ is derived from F. Hence Φ, derived from F by a general linear transformation, differs only in position from F_1, a projection of F; and the general linear transformation, as a process for discovering geometrical theorems, is not one whit more general than projection. For drawing a figure on a different sheet of paper, or on a different part of the same sheet, tells us nothing new about it.

For example, take the case of a quadrilateral and a square. Draw a square $\alpha\beta\gamma\delta$ of any size anywhere in the plane of the quadrilateral; by rotation and translation this can be brought into perspective with the quadrilateral. Construct for V as before; turn the square about so that $\beta\gamma$ may be parallel to VE (Fig. 45); move the square parallel to itself so that β shall remain on VB; slide it along VB until γ is on VC; then the square is in perspective with the quadrilateral.

Ex. Show that any two quadrilaterals can be placed in perspective.

213. Hence we see that the methods of linear transformation and projection can be connected in one of two ways.

From an equation $F=0$ let a second equation $F'=0$ be derived by a linear transformation. Draw separately the two curves C, C', represented by these two equations. Then

(1) C, C' can be placed in space, or in a plane, so that one is the projection of the other;

(2) placing the two diagrams in the same plane, with the lines x, y, z in the same position, a geometrical construction in the plane can be found by which the points of C' are obtained from the points of C.

But also the two equations $F=0$, $F'=0$ can be regarded as representing the same curve, with a different triangle of reference.

214. Stating the plane construction for projection in the simplest possible way, it is:—

Let O be the centre, so that all lines through O correspond to themselves; let any line through O meet the line that is to go to infinity in N; let NP meet the axis in J, through J draw a parallel to ON meeting OP in P'; then P' is the projection of P (Fig. 50). The line taken for ON may be any line through O; the same point P' will in any case be found. It is often convenient to take the axis to pass through O.

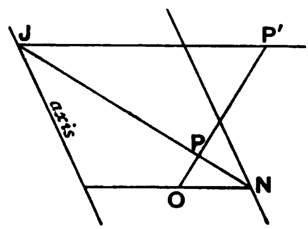

FIG. 50.

The assumption that all lines through O can be made self-correspondents is justified by the possibilities of projection. For let any three lines through O cut the line that projects to infinity in X, Y, Z; we can then, by the general theorem of § 200, project so that the angles XOY, YOZ retain their values, while the assigned line goes to infinity. Let OW be any fourth line; the pencil $\{O.XYZW\}$ is unalterable by projection; and consequently the angle ZOW is unaltered.

215. This construction enables us to determine the formulæ of transformation more simply than by the process of § 209. Let the line that is to be projected to infinity, and the line that is taken as axis, be

$$ax+by=1, \quad ax+by=c.$$

Take the axis of x for ON; let P be x, y and use X, Y for current coordinates. The equations of the lines of the diagram, in the order used in the construction of § 214, are;

$$NP, \quad \frac{X-\frac{1}{a}}{x-\frac{1}{a}}=\frac{Y}{y};$$

$$JP', \quad Y(ax+by-1)=y(c-1);$$

$$OP, \quad \frac{X}{x}=\frac{Y}{y}:$$

the last two give the coordinates of P'; writing x', y' for these, the relations are

$$x'=\frac{x(c-1)}{ax+by-1},\quad y'=\frac{y(c-1)}{ax+by-1};$$

and
$$x=\frac{x'}{ax'+by'-c+1},\quad y=\frac{y'}{ax'+by'-c+1};$$

if the axis be taken to pass through O, $c=0$, and the relations become

$$x=\frac{x'}{ax'+by'+1},\quad y=\frac{y'}{ax'+by'+1}.$$

Thus in the example of § 209,

$$x=\frac{x'}{x'+y'+1},\qquad y=\frac{y'}{x'+y'+1}.$$

Again, to project $3x+3y+1=0$ to infinity, we apply the transformation

$$x=\frac{x'}{-3x'-3y'+1},\quad y=\frac{y'}{-3x'-3y'+1};$$

and to project $x=1$ to infinity,

$$x=\frac{x'}{x'+1},\qquad y=\frac{y'}{x'+1}.$$

216. It is important to notice that a Cartesian transformation is limited by the condition that the line infinity is to be unaltered. Changing to homogeneous coordinates as directed in § 30, this is expressed by $z'=\lambda z$. Hence we have only six constants at our disposal, two involved in the determination of each new axis, one for the multiplier implied in each coordinate; though in the strict Cartesian use of coordinates, these implied multipliers are not available. If the transformation be further limited by the condition that it is to be orthogonal,—that is, that the axes, being initially at right angles, are to remain so,—one degree of choice is destroyed, and we have three constants at our disposal. The condition that the transformation is an orthogonal Cartesian transformation may be expressed in the form—the circular points are to be unchanged. For they are the intersections of lines parallel to $x^2+y^2=0$, and it is known that this equation is unchanged if the rectangular axes be turned through any angle.

Canonical Forms.

217. One object of projection is to reduce the figure to its simplest form, a conic to a circle, a quadrilateral to a square, harmonic section to bisection, etc. Similarly one object of linear transformation is to reduce an equation to

the most manageable form; that is, to choose the best triangle for fundamental triangle, and the best system of coordinates.

To tell à priori whether one equation can be reduced to another, we must compare the number of disposable constants in the two. For example, using Cartesians, the general equation of the second degree contains five constants; this same number is contained in each of the forms

$$(x-a)^2+(y-\beta)^2=(px+qy+r)^2 \quad\text{..............(1)},$$

$$\{(x-a)^2+(y-\beta)^2\}^{\frac{1}{2}}+\{(x-a')^2+(y-\beta')^2\}^{\frac{1}{2}}=k \quad\text{......(2)};$$

but the forms

$$(ax+by)^2=cx+dy+e \quad\text{......................(3)},$$

$$(ax+by+c)(a'x+b'y+c')=0 \quad\text{..................(4)},$$

contain only four disposable constants; and

$$(x-a)^2+(y-\beta)^2=r^2 \quad\text{......................(5)},$$

$$(x-a)^2+(y-\beta)^2=ax+by+c \quad\text{..................(6)},$$

contain only three. Hence the general equation of the second degree can be written in the form (1), which gives a proof of the focus and directrix properties; and it can be written in the form (2), which shows that the sum or difference of the focal distances is constant; but it cannot be written in the other forms.

218. In homogeneous coordinates the question presents itself in a slightly disguised form; the disposable constants are involved in the implied change of the triangle of reference. When we say that $F(x, y, z)=0$ can be written in the form $\Phi(x, y, z)=0$ we mean that a linear transformation

$$x=l_1x'+m_1y'+n_1z',$$
$$y=l_2x'+m_2y'+n_2z',$$
$$z=l_3x'+m_3y'+n_3z',$$

can be found by means of which $F(x, y, z)=0$ becomes $\Phi(x', y', z')=0$, that is, dropping the accents, $\Phi(x, y, z)=0$. In this eight constants are involved, and the question is whether these can be determined so as to change F into Φ.

For example, $x^2=yz$ is a perfectly general form for a conic. For this is

$$(l_1x+m_1y+n_1z)^2=(l_2x+m_2y+n_2z)(l_3x+m_3y+n_3z),$$

that is,

$$A(x+\lambda y+\mu z)^2=(x+\lambda'y+\mu'z)(x+\lambda''y+\mu''z),$$

and it therefore contains seven disposable constants where-

with to satisfy the five equations obtained by comparing it with any given equation of the second degree. The reduction can therefore be accomplished in a doubly infinite number of ways, as is apparent geometrically, all that is necessary being to take two tangents for the lines y, z, and their chord of contact for the line x.

Again, the constants implied in

$$x^2+y^2+z^2=0$$

are involved in such a way as to give eight disposable constants. Hence the reduction can be accomplished in a triply infinite number of ways; it is accomplished by taking any self-conjugate triangle for triangle of reference.

The general equation of a cubic contains ten terms, therefore nine determinable constants. Now the equation

$$x^3+y^3+z^3+pxyz=0,$$

being equivalent to

$$A(x+\lambda y+\mu z)^3+B(x+\lambda' y+\mu' z)^3+C(x+\lambda'' y+\mu'' z)^3 \\ +D(x+\lambda y+\mu z)(x+\lambda' y+\mu' z)(x+\lambda'' y+\mu'' z)=0,$$

contains nine disposable constants, viz., the six quantities λ, μ, etc., and the three ratios $A:B:C:D$; thus it is a perfectly general form for the cubic.

Another form that contains the right number of constants is

$$px^3+2qx^2z+rxz^2=y^2z.$$

Counting the constants is not an absolutely safe process, for the form of the equations may be such that some are not independent, or some may be inconsistent. But it affords a preliminary test.

219. The simplest form to which an equation, or system of equations, can be reduced without loss of generality by linear transformation is called its canonical form.

Thus for the general equation of the second degree, $x^2+2yz=0$ is really the most reduced form, but the most satisfactory form is the symmetrical one,

$$x^2+y^2+z^2=0.$$

A system of two conics is reduced to its canonical form by taking the common self-conjugate triangle for triangle of reference, when the two equations become

$$ax^2+by^2+cz^2=0, \quad a'x^2+b'y^2+c'z^2=0.$$

For the cubic, the canonical form is

$$x^3+y^3+z^3+6mxyz=0;$$

but this cannot express a proper cubic with a double point. For the simultaneous vanishing of $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ gives

$$8m^3 = -1,$$

and the cubic becomes

$$x^3+y^3+z^3-3\omega xyz=0,$$

where ω is any cube root of unity. But this expression splits up into linear factors.

Thus the canonical form may fail to express certain special forms of the equation, owing to the presence of singularities.

220. As an example of a linear transformation, let the triangle of reference be changed from xyz to the one formed by the lines $x^3+y^3+z^3-3xyz=0$, the equation being

$$x^3+y^3+z^3+6mxyz=0.$$

Note that the three points in which $x=0$ meets the curve are inflexions. For $x=0$ gives $y^3+z^3=0$, that is,

$$(y+z)(y+\omega z)(y+\omega^2 z)=0.$$

Consider any one of these points, $0, \omega, -1$. The ordinary formula for the tangent gives

$$-2m\omega x+\omega^2 y+z=0,$$

that is, $$-2m\omega^2 x+y+\omega z=0,$$

which meets the cubic where $$(y+\omega z)^3=0,$$

that is, at an inflexion. Thus there are nine inflexions, lying by threes on the three lines x, y, z; but only three are real, viz., those lying on the line $x+y+z=0$.

The formulæ of transformation can be written

$$\left.\begin{aligned} x+y+z&=3\lambda x',\\ x+\omega y+\omega^2 z&=3\mu y',\\ x+\omega^2 y+\omega z&=3\nu z', \end{aligned}\right\}\text{..................(i.);}$$

where λ, μ, ν have any values we please, say unity. The equations give

$$\left.\begin{aligned} x&=x'+y'+z',\\ y&=x'+\omega^2 y'+\omega z',\\ z&=x'+\omega y'+\omega^2 z', \end{aligned}\right\}\text{..................(ii.).}$$

Now $x^3+y^3+z^3-3xyz$ becomes $27x'y'z'$, by multiplying together equations (i.); and by multiplying together equations (ii.), xyz becomes $x'^3+y'^3+z'^3-3x'y'z'$.

The given equation can be written

$$x^3+y^3+z^3-3xyz+(6m+3)xyz=0,$$

and the transformed equation is therefore

$$27x'y'z' + 3(2m+1)(x'^3 + y'^3 + z'^3 - 3x'y'z') = 0,$$

that is $$x'^3 + y'^3 + z'^3 + 6\frac{1-m}{1+2m}x'y'z' = 0.$$

Hence, just as before, the nine inflexions lie by threes on the lines x', y', z'; that is, on the lines

$$x + y + z = 0,$$
$$x + \omega y + \omega^2 z = 0,$$
$$x + \omega^2 y + \omega z = 0.$$

Ex. Find what the equation

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

becomes, when the triangle of reference is that formed by the lines

$$-2mx + y + z = 0, \quad x - 2my + z = 0, \quad x + y - 2mz = 0.$$

CHAPTER XI.

THEORY OF CORRESPONDENCE.

Special Cases of (1, 1) *Correspondence.*

221. The formulæ for projection do not depend on any special figure that is to be projected; corresponding to any point P in one plane there is a point P' in the other plane, obtained by means of a line VP through a fixed point V; and in the limiting case when the two planes coincide there is a construction, depending on a fixed point and a fixed line, by means of which P' is derived from P. Now a correspondence of this nature, in which one point of one system corresponds to one point of another, has been considered already (§§ 160 etc.) under the heading homography. A (1, 1) correspondence between the elements of two one-dimensional spaces is homographic; and in the theory of projection and linear transformation we have the extension of the idea of homography to two-dimensional spaces; and similarly a homographic correspondence can be introduced between the elements of two a-dimensional spaces. Moreover, the elements compared need not be of the same nature; the two-fold infinity of points in a plane may be associated with the two-fold infinity of points, or of lines, in a plane; or with the conics of a net, or in general with the elements of any two-dimensional space.

222. We shall now consider certain cases of (1, 1) correspondence between two planes, and shall in general suppose the planes superimposed. Projection, that is, a specialized linear transformation in point or line coordinates, institutes a correspondence of point to point, and straight line to straight line; this is a special case of the general (1, 1) linear correspondence, the Collineation of Möbius (*Der barycentrische Calcul*, 1827; *Werke*, t. i., p. 266). A second case that we shall consider is that where point corresponds to point, but a straight line to a conic; now the number of straight lines in

a plane is doubly infinite, and therefore the conics considered will form a doubly infinite system; we shall find that they all pass through three fixed points. This (1, 1) quadric correspondence will be investigated by means of the theory of Quadric Inversion; we shall find that the formulæ of transformation are of the second degree. And finally, we shall consider the dualistic transformation, in which by an interchange of point and line coordinates a correspondence is instituted between the points of a plane and the lines of a plane; the characteristic part of this dualistic transformation will be found to depend on the theory of Reciprocation.

Collineation.

223. There is a (1, 1) correspondence between the points, and also between the lines, of the two superimposed planes. But exactly as in the case of one-dimensional homography, this does not give a correspondence between the elements of one plane. For let the point of the first plane that comes at A be P, and let the point of the second plane at A be Q'; for simplicity suppose the correspondence to be perspective; P' is then found by the construction of § 214, and reversing this construction, Q is found; P' and Q do not ordinarily coincide. But for a special position of the axis, they do coincide. The formulæ for projective transformation (§ 215) are

$$x'=\frac{x(c-1)}{ax+by-1},\quad y'=\frac{y(c-1)}{ax+by-1};$$

$$x=\frac{x'}{ax'+by'-c+1},\quad y=\frac{y'}{ax'+by'-c+1};$$

and in order that P' and Q may coincide for all positions of A, we must have for all values of x, y

$$\frac{x(c-1)}{ax+by-1}=\frac{x}{ax+by-c+1},$$

$$\frac{y(c-1)}{ax+by-1}=\frac{y}{ax+by-c+1};$$

that is, $$c=2.$$

Thus the line that is to go to infinity being

$$ax+by=1,$$

we are to take as axis

$$ax+by=2.$$

Let ON meet this line in Z; then OZ is bisected in N, that is, $(OZ, N\infty)$ is harmonic. Hence considering the com-

plete quadrilateral formed by the two line-pairs PP', JJ' : PJ', $P'J$ (Fig. 51), we see that (OR, PP') is harmonic. Hence P' can be constructed as the harmonic conjugate to P with regard to O and the intersection of OP with the axis. This special projection is the *harmonic transformation;* and the two figures which are in perspective by this reversible construction are in *involution-position* (*involutorische Lage*).

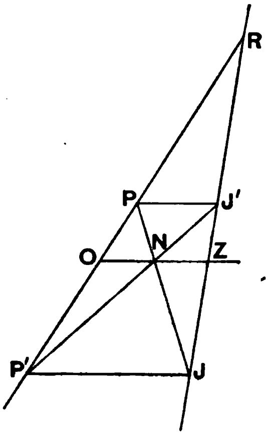

Fig. 51.

As an example of harmonic transformation, let P describe a circle whose centre is O; then P' describes a conic whose focus is O, the directrix being the line through N parallel to the axis. The proof of this by elementary geometry is perfectly simple; the proof by modern geometry is here given. The circle cuts the present line infinity at the circular points ω, ω', and $O\omega$, $O\omega'$ are tangents, $\omega\omega'$ being the chord of contact. Lines through O transform into themselves, and the line $\omega\omega'$ transforms into the line through N parallel to the axis; hence though the circular points are changed, the isotropic lines through O are unchanged. By the linear transformation in question, the circle becomes a conic; O is the intersection of isotropic tangents, and is therefore a focus; and the specified line through N, being the polar of the focus O, is the directrix.

We have here seen that the points of a plane *can* be associated in pairs. When the elements of any two-dimensional space are associated in pairs, they may be said to be in involution; and, exactly as in one-dimensional geometry, involution can be regarded as a special case of homography by the device of counting every element of the space twice, though this disguises the fundamental idea in involution.

224. The harmonic transformation, just given, is the only linear transformation by which the points of a plane can be arranged in involution. For without making the assumption that two figures are in perspective, let AA', BB', etc., be pairs of correspondents. The transformation being linear, the line AA' corresponds to itself as a whole, and similarly for BB'; hence the correspondent to O, the intersection of these lines, must lie on AA' and on BB'; O is therefore its own correspondent. It is consequently a double point of the involution on AA', and of the involution on BB'; and by the harmonic properties of double points, the other self-correspondents on AA' and BB' are constructed by means of a diagonal of the quadrangle $AA'BB'$. Let them be X, Y: on

this diagonal there is an involution, with X, Y as double points; and the line corresponds to itself as a whole. Hence the point in which XY meets AB corresponds to the point in which XY meets $A'B'$. But these points coincide at C (Fig. 52). Thus in the involution on XY one point, other than the double points, coincides with its correspondent, and therefore every point on the line is its own correspondent, and consequently every line through O is its own correspondent. To construct the correspondent to any point P, let OP meet XY in R; O, R are the double points of the

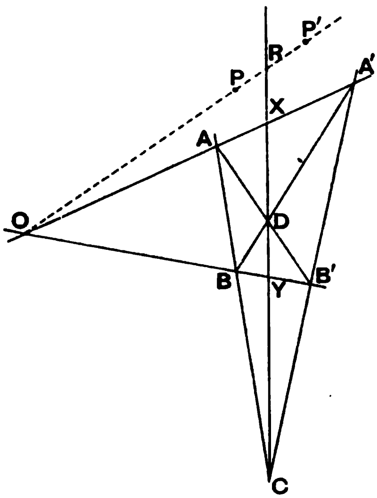

FIG. 52.

involution on OP, in which PP' are correspondents; hence (OR, PP') is harmonic, and the points of the plane are connected by the harmonic transformation, as are also the lines of the plane.

225. From § 223 we see that if it be known that two figures are in perspective, then the fact that one pair of points are conjugates proves that the figures are in involution; and § 224 shows that if it be not known that the figures are in perspective, but only that there is between them a (1, 1) linear correspondence, then the fact that two pairs of points are conjugates proves that the figures are in involution.

In one-dimensional geometry, two homographic ranges can always be placed so as to be in involution; but there is no corresponding theorem in the present case. Homographic figures can be placed in perspective, but we have seen that figures in perspective are not necessarily in involution.

226. In homographic systems in the same one-dimensional space there are two elements that coincide with their correspondents. We now consider what elements are self-correspondent in the general linear transformation.

Since both point and line coordinates are subject to linear transformation (§ 34), we must expect a certain number of points and the same number of lines to be self-correspondent. Now if A, B correspond to A', B', the line AB as a whole corresponds to $A'B'$ as a whole; thus the self-corresponding lines are obtained by joining the self-corresponding points in pairs. Let A, B be two self-corresponding points; on AB we have two homographic ranges with A, B as double points; hence on the line AB there are ordinarily no more self-corresponding points, that is, not more than two points on a line or two lines through a point can be self-correspondent. Hence if there be any self-corresponding points and lines, there must ordinarily be three of each, these being the vertices and sides of a triangle ABC; two of the points, situated on a real line, may be imaginary, and then the sides opposite to them are imaginary with a real intersection.

But now suppose that a third point on the line AB corresponds to itself; then every point on AB corresponds to itself. If there be no self-corresponding point off this line, the triangle ABC of the general case has its three sides coincident. If there be a self-corresponding point C off the line, then every line through C corresponds to itself; the sides a, b and the vertices A, B of the triangle ABC are indeterminate, but the side c and the vertex C are determinate; this last case is simply projection.

The fact that there *are* self-corresponding elements appears at once from the equations for linear transformation. These give

$$x : y : z = l_1x' + m_1y' + n_1z' : l_2x' + m_2y' + n_2z' : l_3x' + m_3y' + n_3z',$$

and we are to have

$$x : y : z = x' : y' : z';$$

therefore

$$\frac{l_1x + m_1y + n_1z}{x} = \frac{l_2x + m_2y + n_2z}{y} = \frac{l_3x + m_3y + n_3z}{z} = \lambda,$$

that is,

$$\begin{aligned} (l_1 - \lambda)x + \quad m_1y + \quad n_1z &= 0, \\ l_2x + (m_2 - \lambda)y + \quad n_2z &= 0, \\ l_3x + \quad m_3y + (n_3 - \lambda)z &= 0, \end{aligned}$$

from which by eliminating x, y, z a cubic equation is obtained for λ. The three roots, of which one is necessarily real, give

three sets of values for x, y, z, and therefore three self-corresponding points A, B, C. Taking the triangle ABC for triangle of reference, $x=0$ corresponds to $x'=0$, etc., hence the formulæ of transformation reduce to

$$x=fx', \quad y=gy', \quad z=hz'.$$

But if the cubic in λ have two, or three, equal roots, a certain degree of indeterminateness is introduced. To illustrate this point, consider the transformation

$$x=x'+az', \quad y=y'+bz', \quad z=z'.$$

The cubic in λ has its three roots equal; the three points ABC are indeterminate, all lying on $z=0$.

Again, consider the projective transformation

$$\begin{aligned} x&=kx'+y'+z', \\ y&=x'+ky'+z', \\ z&=x'+y'+kz'. \end{aligned}$$

The cubic in λ has now two equal roots, $k-1$, the third root being $k+2$. The non-repeated root gives a definite point C $(1, 1, 1)$; the repeated root gives simply $x+y+z=0$, the line c.

Examples.

1. Show that the effect of rotation on a figure can be represented by a linear transformation, in which the three fixed points are the circular points and the centre of rotation.

2. Discuss the case of translation.

3. Show that in any rotation the line infinity has its points displaced along itself; and that in any translation the line infinity is entirely unaltered.

4. In the case of harmonic transformation, if the axis be at infinity, the centre is a centre of symmetry; and if the centre be at infinity, the axis is an axis of symmetry.

General Theory of Correspondence.

227. The general theory of correspondence may be presented in a slightly different way. Let there be a system of elements, forming a p-fold infinity; any one of these may be indicated by the ratios of $p+1$ parameters; calling these $\lambda, \mu, \nu, \ldots, \sigma$, the element is $\lambda, \mu, \nu, \ldots, \sigma$. Taking any other p-fold infinity, the element $\lambda, \mu, \nu, \ldots, \sigma$ of one system can be regarded as corresponding to the element $\lambda, \mu, \nu, \ldots, \sigma$ of the other. For example, let S_1, S_2, S_3 be conics; consider the net

$$\lambda S_1+\mu S_2+\nu S_3=0;$$

in this any conic can be regarded as the element λ, μ, ν; it can be considered as corresponding to a point λ, μ, ν, or to a line

$$\lambda x + \mu y + \nu z = 0,$$

or more generally to a curve

$$\lambda F_1 + \mu F_2 + \nu F_3 = 0.$$

This view of correspondence is somewhat disguised in the ordinary presentation of projection; it appears more plainly in the interchange of point and line coordinates. In this the marks λ, μ, ν are attached on the one hand to a point, on the other hand to a line, as coordinates or as coefficients.

The algebraic statement of a theorem relating to one set of elements with characteristics $\lambda, \mu, \nu, \ldots, \sigma$ may be exactly the same as for a totally different set of elements with the same characteristics (compare § 53); we have then a means of generalizing. From a geometrical theorem relating, for example, to lines

$$\lambda x + \mu y + \nu z = 0,$$

we pass to a theorem relating to conics

$$\lambda S_1 + \mu S_2 + \nu S_3 = 0.$$

This transition is not by means of deduction, but it depends on an implied reference to a second interpretation of the algebraic work by which we know that the theorem can be proved, even though the proof actually given may be geometrical.

228. But we can regard this principle of correspondence in a more exclusively geometrical way, without even this implied reference to analysis. A figure in any space is generated by a moving element coinciding successively with certain of the ultimate atoms of the space, according to some law of selection. Now let there be another space, with ultimate atoms of a different kind (or of the same kind), the order of the manifoldness being however the same; and imagine a moving element to coincide successively with certain of these, the law of selection being the same as before. The behaviour of the second generating element and the laws of the second configuration can to a certain extent be deduced from the behaviour of the first generating element and the laws of the first configuration; theorems proved for the first configuration afford theorems relating to the second configuration.

The practical difficulty that presents itself here is in the expression of the law of selection. This must be expressed in terms of the elements of the space. Suppose for example that the correspondence considered is between the points of a

plane and the lines of a plane; and that it is of such a nature that to a line in the first plane corresponds a point in the second; this may be called a linear aggregate of elements. Now let a point describe a conic in the first plane; this can be expressed in terms of the elements of the plane. The moving element coincides successively with a singly infinite number of the fixed elements, these being chosen so that there are two belonging to every linear aggregate. The law of selection, so stated, is referable to the second system, and gives an envelope of the second class. (Compare § 54.)

229. We may if we choose concentrate our attention entirely on the laws and operations manifested in these different geometrical figures; these figures with their related theories are then unimportant transitory incarnations of an underlying unchangeable principle; the differences perceived by the eye are neglected and the figures are regarded as the same, inasmuch as they express the same sequence and connection of elements. From this point of view a curve and all its projections, or a curve and its reciprocal, are essentially identical; a system of lines in a plane may be identical with a system of conics in a net. This view pervades a great deal of recent work; it is formulated by Professor Klein in his address *Vergleichende Betrachtungen über neuere geometrische Forschungen*, 1872*:—"Streifen wir jetzt das mathematisch unwesentliche sinnliche Bild ab.... Aber die projectivische Geometrie erwuchs erst, als man sich gewöhnte, die ursprüngliche Figur mit allen aus ihr projectivisch ableitbaren als wesentlich identisch zu erachten.... Wenn wir im Texte die räumliche Anschauung als etwas Beiläufiges bezeichnen, so ist dies mit Bezug auf den rein mathematischen Inhalt der zu formulirenden Betrachtungen gemeint. Die Anschauung hat für ihn nur den Werth der Veranschaulichung."

We are however here concerned with the manifestations of the underlying principles and operations; these last exist for us only as the cause of the correspondence that we consider.

General (1, 1) *Quadric Correspondence.*

230. Important examples of this deduction of one theorem from another occur in the use of a particular (1, 1) correspondence, geometrically represented by Quadric Inversion. This presents itself as the next simplest case to projection; it is a correspondence of point to point, and of straight line to conic, that is, it is a (1, 1) quadric correspondence.

Since the number of straight lines in a plane is doubly infinite, the conics considered must form a two-fold infinity; the equation of a line being

$$\lambda L_1 + \mu L_2 + \nu L_3 = 0,$$

* For a reprint of this address, see p. 63, t. xliii. of the *Mathematische Annalen;* for a translation, see p. 215, vol. ii. of the *Bulletin of the New York Mathematical Society*.

where L_1, L_2, L_3 are three particular lines corresponding to the conics S_1, S_2, S_3, the corresponding conic is

$$\lambda S_1+\mu S_2+\nu S_3=0.$$

But by the given conditions, as regards points the correspondence is (1, 1); now the intersection of two lines corresponds to the intersection of the corresponding conics; hence of the four intersections of any two conics, all but one must be automatically excluded; three must be fixed, and one variable. Hence all the conics considered must pass through three fixed points. These points ordinarily form a triangle, but there are special cases that may be considered, arising from coincidences. At present we consider only the general case.

Taking this triangle as triangle of reference in the conic-plane, the conics S_1, S_2, S_3 have equations

$$\begin{aligned}f_1yz+g_1zx+h_1xy&=0,\\ f_2yz+g_2zx+h_2xy&=0,\\ f_3yz+g_3zx+h_3xy&=0;\end{aligned}$$

hence to the line $\qquad \lambda L_1+\mu L_2+\nu L_3=0$

there corresponds the conic

$$\lambda S_1+\mu S_2+\nu S_3=0,$$

that is, the conic

$$(\lambda f_1+\mu f_2+\nu f_3)yz+(\lambda g_1+\mu g_2+\nu g_3)zx+(\lambda h_1+\mu h_2+\nu h_3)xy=0.$$

Now change the triangle of reference in the line-plane by means of the formulæ

$$\begin{aligned}L_1&=f_1x'+g_1y'+h_1z',\\ L_2&=f_2x'+g_2y'+h_2z',\\ L_3&=f_3x'+g_3y'+h_3z';\end{aligned}$$

the correspondence is then between the straight line

$$(\lambda f_1+\mu f_2+\nu f_3)x'+(\lambda g_1+\mu g_2+\nu g_3)y'+(\lambda h_1+\mu h_2+\nu h_3)z'=0,$$

and the conic

$$(\lambda f_1+\mu f_2+\nu f_3)yz+(\lambda g_1+\mu g_2+\nu g_3)zx+(\lambda h_1+\mu h_2+\nu h_3)xy=0;$$

hence the transformation is expressed by the equations

$$\begin{aligned}x':y':z'&=yz:zx:xy\\ &=\frac{1}{x}:\frac{1}{y}:\frac{1}{z},\end{aligned}$$

which are the same as

$$x:y:z=\frac{1}{x'}:\frac{1}{y'}:\frac{1}{z'}.$$

These formulæ of transformation can be written

$$xx'=1,\quad yy'=1,\quad zz'=1,$$

that is, under the form of bilinear relations.

We found in § 212 that the general linear transformation differs from projection only in position; and here we have proved that the general reversible quadric transformation differs only by a linear transformation from

$$x:y:z=\frac{1}{x'}:\frac{1}{y'}:\frac{1}{z'}.$$

231. This general quadric correspondence can be produced by a kind of projection in space. Take two non-intersecting straight lines a, b, and two planes π_1, π_2 not containing either line, met by the lines in A_1, A_2, B_1, B_2. Let l be any line in π_1; a, b, l determine a hyperboloid of one sheet, and this is cut by π_2 in a conic; thus the lines in π_1 are projected into conics in π_2. Now whatever line l we take we can draw from A_2 a line to meet b and l; hence the conic projection of every line goes through A_2, and similarly it goes through B_2. Also the line A_1B_1 meets a, b, and l; if therefore A_1B_1 meet π_2 in C_2, C_2 is a point on the conic projection of l. Hence all the conics considered pass through three fixed points A_2, B_2, C_2. Special cases arise owing to special relations among the given fixed elements; for example, if the two given lines a, b meet in a point V, this *skew projection* reduces to the ordinary, or conical projection. The transformation of figures thus obtained is the Steiner transformation; it was fully explained by Steiner in his *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*, Berlin, 1832. *Ges. Werke*, t. i., pp. 409, 421.

Quadric Inversion.

232. We now consider a plane construction by means of which this quadric correspondence can be produced; this was given by Dr. Hirst, in the *Proceedings of the Roydl Society*, 1865. The correspondence of point to point is effected by means of a fixed origin O, and a fixed fundamental conic, the base; points that are collinear with the origin, and conjugate with respect to the base, are said to be inverse; if for the fundamental conic and the origin we take a circle and its centre, the points are the ordinary inverse points with regard to a circle, hence the process is simply circular inversion generalized, that is, it is quadric inversion. As regards points, the correspondence is $(1, 1)$; moreover it is reversible, and it associates the points of the plane in pairs PP'; that is, to a point P of the plane there corresponds definitely a point P'. There are however exceptional elements; for let the tangents from O to the base be OI, OJ; by the definition, P', inverse to P, is the intersection of OP and the polar of P; this is indeterminate

(1) if P be at O; then P' is any point on IJ;

(2) if P be at I; then P' is any point on OI;

(3) if P be at J; then P' is any point on OJ.

Hence if P be at any vertex or on any side of the triangle

OIJ, the ordinary laws of the correspondence are not applicable. For the present these special positions of P are not taken into account.

To consecutive points correspond consecutive points; hence an arc of a curve gives an arc of a curve, and arcs intersecting or touching give arcs intersecting or touching.

Let P_1, P_2, P_3, P_4 be points on a line q, meeting the base in A, B; let the pole of q be Q, and let the polars of P_1 etc. be p_1 etc., so that P_1' is the intersection of OP_1 and p_1. Then

$$\{P_1P_2P_3P_4\} = \{p_1p_2p_3p_4\};$$

now the four lines p_1, p_2, p_3, p_4 form a pencil whose vertex is Q, and P_1', etc. are points on these lines; hence the relation just given can be written

$$\{O\,.\,P_1P_2P_3P_4\} = \{Q\,.\,P_1'P_2'P_3'P_4'\},$$

that is,
$$\{O\,.\,P_1'P_2'P_3'P_4'\} = \{Q\,.\,P_1'P_2'P_3'P_4'\}.$$

This shows that the points P', inverse to collinear points P, lie on a conic through O, Q; and as any point on the base is its own inverse, the points A, B belong to this conic. Let the line q cut OI in X; by the general construction the inverse to X is at I, hence the conic passes through I, and through J. The inverse to any line q is therefore a conic through the three principal points.

Moreover we have here shown that the cross-ratio of the four points on the line is equal to the cross-ratio of the pencil determined in the inverse conic by the inverse points. Hence points in involution on a straight line give rise to points in involution on the inverse conic; the transformation is in fact homographic.

Note. O is the origin, necessarily real; I, J are the fundamental points, they may be real or imaginary; O, I, J are the principal points. The three sides of this triangle are the principal lines, and of these OI, OJ may be distinguished as the fundamental lines (Hirst).

233. Hence the correspondence is that of straight line to conic, and the conics satisfy the condition of § 230. Take OI, OJ, IJ for the sides x, y, z of the triangle of reference, and let the coordinates be chosen so that the fundamental conic is

$$z^2 - xy = 0.$$

Let P' be x', y', z'; the polar of P' is

$$-xy' - yx' + 2zz' = 0,$$

and the equation of OP' is

$$xy' - yx' = 0;$$

hence the coordinates of P are given by

$$x : y : z = x' : y' : \frac{x'y'}{z'},$$

$$= x'z' : y'z' : x'y'.$$

Applying to x', y', z' a linear transformation

$$x' : y' : z' = y_1 : x_1 : z_1,$$

the formulæ become

$$x : y : z = \frac{1}{x_1} : \frac{1}{y_1} : \frac{1}{z_1} :$$

hence the geometrical transformation now under investigation gives the general (1, 1) quadric correspondence of § 230.

234. In the analytical theory special cases arise owing to special positions of the three principal points;

(i.) the three may be distinct:
(ii.) two may come together:
(iii.) the three may come together.

In the geometrical theory, we naturally discriminate according as the fundamental conic is proper or degenerate, and according as it does not or does pass through O. Hence four cases present themselves;

(1) the base may be a proper conic not passing through the origin;

(2) the base may be a degenerate conic, not passing through the origin;

(3) the base may be a proper conic passing through the origin;

(4) the base may be a degenerate conic, composed of two straight lines, of which

(*a*) one passes through the origin; this is simply harmonic transformation (§ 223); for the base being JO, JV, the inverse to P is the intersection of OP with the polar of P; let OP meet JV at V, then (OV, PP') is harmonic;

(*b*) both pass through the origin; the transformation becomes indeterminate, for if P be at O, P' is any point in the plane; and if P be not at O, P' is at O.

The first three cases correspond in order to the three cases in the analytical theory. For in (1), the three points O, I, J are distinct; in (2), the points I, J coincide, as may be seen

by considering a hyperbola on the point of degenerating into two straight lines; and in (3), the points I, O, J are consecutive points on the fundamental conic, a fact which is made evident by a diagram showing tangents OI, OJ drawn to a conic from a point O just off the conic. The equations for (1) are given in § 233; to deal with case (2), in which I, J coincide, take for the sides x, z, y, the line OJ, the polar of O, and any arbitrary line through O. The fundamental conic is a pair of lines through xz, harmonic with respect to x, z; hence by a proper choice of coordinates its equation is made to be

$$x^2 - z^2 = 0.$$

The polar of $P'(x', y', z')$ is

$$x'x - z'z = 0,$$

and the equation of OP' is

$$y'x - x'y = 0,$$

hence the point P is given by

$$x : y : z = x' : y' : \frac{x'^2}{z'}$$

$$= x'z' : y'z' : x'^2;$$

from which $\quad x' : y' : z' = xz : yz : x^2.$

In case (3), taking for x, y, z the tangent at O, any chord through O, and the tangent at the other point where this chord meets the conic, the equation of the conic is of the form

$$ky^2 - 2xz = 0,$$

where k is at our disposal; if P' be x', y', z', the polar of P' is

$$z'x - ky'y + x'z = 0,$$

and the line OP' is

$$y'x - x'y = 0.$$

Hence the point P is given by

$$x : y : z = x'^2 : x'y' : ky'^2 - x'z',$$

from which $\quad x' : y' : z' = x^2 : xy : ky^2 - xz.$

235. We have seen that in general the inverse to a straight line is a conic through O, I, J. But if the straight line pass through O, then the pole Q is on IJ, and the conic $OIJQAB$ has three points on the line IJ; it is therefore degenerate, composed of IJ and the given line q. IJ presents itself here simply because the inverse to O is indeterminate, being any point on IJ; and similarly whenever a curve to be inverted passes through O, the line IJ presents itself as part

of the inverse. Also a line through I, meeting the base again in H, has for its inverse the degenerate conic composed of OI and JH; for the point Q is now on OI, hence the points O, I, Q on the conic are accounted for by the line OI, and the remaining points J, H give the other line of the degenerate conic; and similarly if a curve pass through I (or J) the line OI (or OJ) presents itself as part of the inverse, this being necessitated by the fact that the point inverse to I is indeterminate, being any point on OI. These factors thus occurring in the inverse are not counted as part of the proper inverse; they are rejected, and only the residual inverse is counted. Thus for example if a conic through OIJ cut the fundamental conic again in AB, the formulæ of transformation give for the inverse the four lines OI, OJ, IJ, AB; the proper inverse is simply the line AB.

The conic is $$fyz+gzx+hxy=0,$$
and the inverse is $$fx'y'^2z'+gx'^2y'z'+hx'y'z'^2=0,$$
that is, $$x'y'z'(fy'+gx'+hz')=0.$$

236. The fact that lines through I, J intersecting on the base are inverse affords the most generally convenient way of determining graphically the inverse to a point P when I, J are distinct. Let IP meet the base in H, then JH meets OP in P'. By this means we can divide the plane into pairs of inverse compartments 11′, 22′, etc., and when a point P passes from 1 to 2, the inverse point P' passes from 1′ to 2′.

To see clearly the effect of inversion on a curve, it is advisable to draw the conic that corresponds to the line infinity. The point Q of § 232 is now C, the centre of the fundamental conic; the points A, B are the points at infinity on the fundamental conic. The point at infinity on the line IJ being Ω, the polar of Ω passes through O, and the line $O\Omega$ is parallel to IJ, hence the inverse to Ω is consecutive to O on the line $O\Omega$; that is, the tangent at O to the required conic is parallel to IJ. Hence OC is the diameter conjugate to IJ, and the tangent at C is parallel to IJ; these conditions are more than sufficient to determine the conic, but it may also be noticed that drawing from J a line parallel to OI to meet the base in K, IK is the tangent at I, and similarly the tangent at J can be constructed. In Fig. 53 the conic inverse to infinity is represented by a broken line.

Note. One advantage of this division into compartments is that in the next few diagrams there is no occasion to represent the fundamental conic. It is advisable to consider the arrangement of the compartments not only for the ellipse, shown in Fig. 53, but also for the hyperbola, and

then, making I, J approach one another indefinitely, for the line-pair. It appears further on that for one important use to which the method of quadric inversion is put, the line-pair is the best fundamental conic.

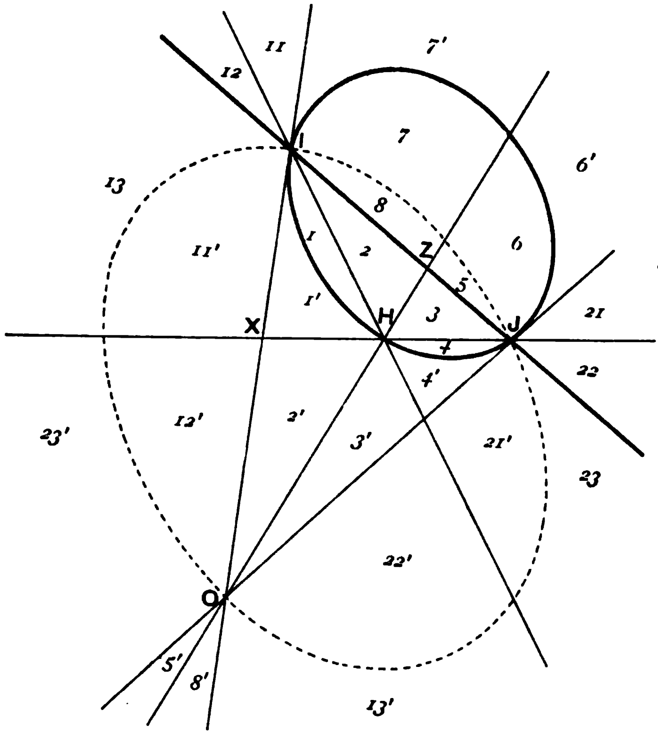

FIG. 53.

Effect of Inversion on Singularities.

237. The law of construction shows that an ordinary point inverts into a single point; but this may be an inflexion. For at the ordinary point three consecutive points are not collinear; but if these three points and O, I, J lie on a conic, then their inverses are collinear, and on the inverse curve there is an inflexion. Similarly an inflexion may be lost by inversion; it will be lost unless the inflexional tangent pass through a principal point.

In the same way, while a double point inverts into a double point of the same nature as regards the division of the tangents into real and imaginary, and adjacent double points into adjacent double points, the appearance may be slightly altered. Two branches having ordinary contact give two adjacent double points, forming a tacnode, and the inverse

has a tacnode; but two branches having closer contact (contact of the second order) intersect in three consecutive points, and cross each other; these three intersections give three adjacent double points, and the singularity (being derived from osculating branches) is an oscnode, straight or curved according as the three nodes are collinear or not; plainly at a straight oscnode, each branch has an inflexion. Now the three consecutive nodes, being initially collinear, will lose this property on inversion, and the straight oscnode will become curved, unless the tangent pass through a principal point; and on the other hand the curved oscnode may become straight. Similarly double tangents may be gained or lost by inversion; and the general conclusion is that as regards points and lines that do not belong to the triangle OIJ, point-singularities are the same on the curve and its inverse, but line-singularities are altered.

238. But the case is very different when we deal with points on the principal lines. Let the curve cut IJ at Z, passing from 2 to 5 (Fig. 53); the inverse to Z is on OZ, by definition, and as the polar of Z passes through O, the inverse is indefinitely near to O, on OZ; consequently OZ is the tangent at O, which enables the inverse to pass from $2'$ to $5'$, as it must by § 236; O is an ordinary point on the curve, having OZ as tangent. If then the curve cut IJ at n points, other than I, J, the inverse has at O a multiple point of order n; for example, an ordinary branch cutting IJ at Z, Z' (Fig. 54 (a)) gives rise to a loop with a node at O, the two tangents being OZ, OZ'. If however OZ be the tangent at Z, so that the curve passes from 2 to 8 (Fig. 53), the inverse has OZ as an *inflexional* tangent at O, and passes from $2'$ to $8'$.

239. Now let ZZ' coincide, the segment ZTZ' vanishing; that is, let there be a branch having contact with IJ at Z; the points ZZ' are properly consecutive, not coincident. We still have at O a double point, on account of the two points Z on IJ; this is the limit of the loop with the node, when the two tangents close up and the loop disappears; the original branch being in 2, 3, the inverse is in $2'$, $3'$, with OZ as tangent to the two parts; and we see that the inverse has a cusp at O. Moreover, since the two points Z, Z' are consecutive, not coincident, the two tangents at the cusp, OZ, OZ', are consecutive, not coincident; and again, since O is a double point by means of the inverses to consecutive points Z, Z', which property of consecutiveness is not destroyed by inversion, it follows that a cusp is produced by the co-

incidence of consecutive points, while a node is produced by the coincidence of non-consecutive points (Fig. 54 (*a*)).

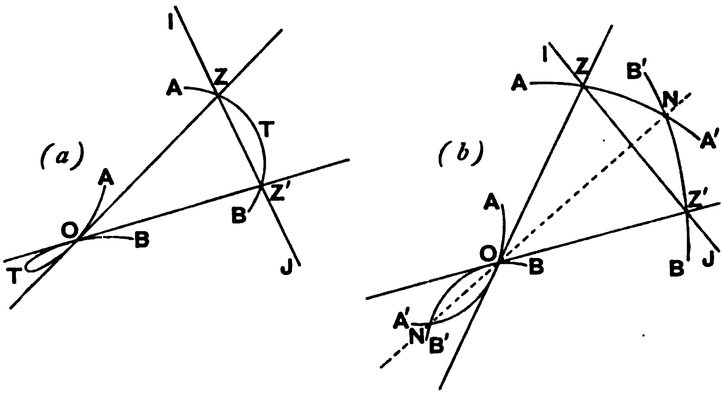

FIG. 54.

The coincidence of Z, Z' may however be brought about without contact with IJ, by means of a node or cusp at Z. A node is caused simply by the crossing of two ordinary branches at Z, and the inverse therefore exhibits the contact of two ordinary branches at O; it has a tacnode, in which the two branches have their concavities in opposite directions, if the two intersecting branches in the original occupy different pairs of compartments, 2, 5 and 3, 8 in Fig. 53; but in the same direction if the intersecting branches occupy the same pair of compartments. The constitution of a tacnode is shown by Fig. 54 (*b*); the node N is shown just off the line IJ, and on the inverse there are two nodes, O, N; as the node approaches IJ indefinitely, Z, Z' becoming coincident, these two nodes become consecutive on the line OZ, which is ultimately the common tangent to the two branches.

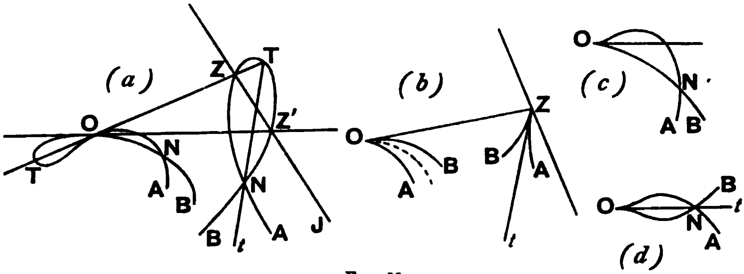

FIG. 55.

To investigate the nature of the singularity at O due to a cusp at Z, we replace the cusp by its penultimate form

(Fig. 55 (a)); inverting this, there is at O an evanescent loop, followed by a node N, not on the tangent OZ (Fig. 55 (c)). The appearance is that of the cusp of the second species (Fig. 55 (b)), the tangent t to the original cusp inverts into a conic, separating the two branches that form the cusp in the inverse, these branches now curving in the same direction. If however t pass through O, the corresponding diagrams show that the appearance is that of the ordinary cusp, but that there is in reality a cusp, followed by a node N lying on the cuspidal tangent (Fig. 55 (d)).

240. Now let three points Z, Z', Z'' come together; this may be due to inflexional contact with IJ, the branch of the curve lying, for example, in 2 and 5. The branch of the inverse being in 2′ and 5′ does not differ in appearance from an ordinary branch; but there is really a triple point at O. The diagram with the three Z's separated shows that the triple point has the three tangents consecutive, not coincident, and contains two evanescent loops, as represented in Fig. 56 (a).

The coincidence of three Z's, again, may be due to a node, having IJ for one tangent; the inverse exhibits an ordinary branch and a cusp, the two having the same tangent.

It may be due to a cusp, with IJ as tangent; let this be in 3, 5; then the inverse is in 3′, 5′, with OZ as tangent, and therefore it presents the appearance of an ordinary inflexion; but drawing the diagram for the penultimate form of the cusp, with the node N on the line IJ, the triple point involved in this apparent inflexion will be made evident, with two evanescent loops (Fig. 56 (b)). It will also be seen that of the

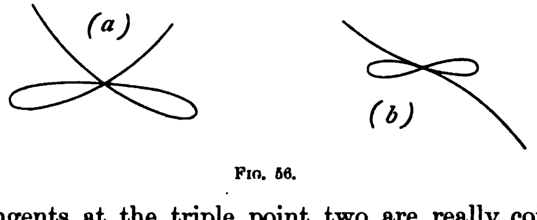

FIG. 56.

three tangents at the triple point, two are really coincident, and the third is consecutive to these.

Finally, the coincidence of three Z's may be due to a triple point at Z; but the figures for the various cases are all constructed in the same way, and therefore no more need be given. We pass on to consider analytically what has just been done.

241. Having a singular point at O, with OZ as the only

tangent, the inverse to this is found at Z. It will be convenient therefore to take OZ as a side of the triangle of reference, if possible. But if I, J be distinct, the sides of the triangle of reference must be OI, OJ, IJ, in order to use the formulæ of transformation found in § 233; if however I, J be coincident (§ 234) only two sides of the triangle of reference are determined, and the third side, y, is any arbitrary line through O; we can therefore take for it the tangent at the singularity to be examined. If any of the tangents at the singularity be different from OZ, only a part of the inverse will be found at Z; the rest will be at Z', Z'', etc., these being the points on IJ determined by the tangents distinct from OZ. The formulæ of transformation are

$$x : y : z = x'z' : y'z' : x'^2.$$

Let the singularity to be examined be that at the point xy on the curve

$$y^3 z = x^4.$$

The inverse to this is

$$x'^2 y'^3 z'^3 = x'^4 z'^4,$$

but the factors x'^2, z'^3 are to be rejected to obtain the proper inverse, which is therefore

$$y'^3 = x'^2 z'.$$

The point Z is $y'z'$; the nature of the curve at Z is therefore found by writing $x' = 1$, which gives

$$y'^3 = z'.$$

Hence there is an inflexion at Z, with IJ as tangent; and the singularity at O on the original curve is the triple point whose penultimate form is shown in Fig. 56 (a).

242. Instead of using the two sets of homogeneous coordinates, x, y, z with $z = 1$, for the singularity at O, and x', y', z' with $x' = 1$, for its inverse at Z (Fig. 57), it is often more convenient to write for x', y', z', new coordinates 1, y_1, x_1, which of course simply amounts to changing the names of the sides of the triangle of reference. The formulæ of transformation are then

$$x = x_1, \quad y = x_1 y_1.$$

For example, let it be required to determine the nature of the singularity at O on the curve

$$y^3 = x^5.$$

The inverse is $$x_1^3 y_1^3 = x_1^5,$$

and therefore the proper inverse is

$$y_1^3 = x_1^2.$$

This has at x_1, y_1 a cusp with x_1, that is, IJ, for tangent. Hence the singularity at O (Fig. 57) is the one whose penultimate form is shown in Fig. 56 (*b*).

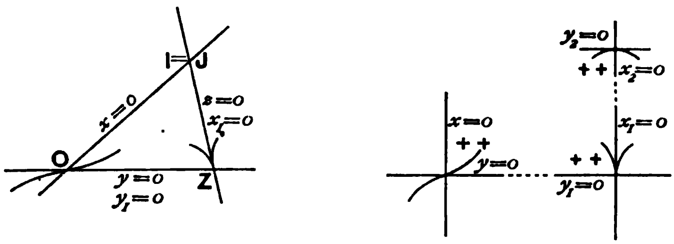

FIG. 57.

Since a single inversion may replace the singularity by a singularity whose penultimate form is not known, it may be necessary to resolve this second singularity by inversion. For instance, in the example just used, we find $y_1^3=x_1^2$. This has a singularity at x_1y_1, with $x_1=0$ for tangent; hence the formulæ of transformation for this case are

$$y_1=y_2, \quad x_1=x_2y_2,$$

and the *second inverse* is

$$y_2^3=x_2^2y_2^2,$$

that is,

$$y_2=x_2^2.$$

Thus by two inversions the arc with the singularity is reduced to an ordinary arc. The steps are represented in the right-hand part of Fig. 57.

Note. The formulæ of transformation are applied without any explicit reference to the fundamental (degenerate) conic; in the example just used, the two inversions are accomplished by means of two different conics. When a number of inversions have to be performed, it is convenient to represent them as in the right-hand part of Fig. 57, not using the triangle at all. In passing from one set of axes to another, care should be taken in noticing the positive sides of the new axes; and it must be noticed that the formulæ depend on the tangent; $u=0$ being the tangent, and $v=0$ any other line through the point considered, the formulæ are

$$v=v', \quad u=u'v'.$$

243. The transformation here explained is the ordinary quadric transformation employed in investigations in the theory of algebraic functions; it is used in the form just given, for purely algebraic purposes, in papers by Brill and Nöther in the *Mathematische Annalen*, tt. vii., ix., xxiii., etc. It was first used by Newton in his *Enumeratio Linearum Tertii Ordinis*, 1704; the curve obtained is called a hyperbolism of the original curve, and it is the whole curve that is considered, not any special point. The same transformation is used by Cramer in his *Analyse des Lignes Courbes*, 1750, for the analysis of singularities; but the geometrical connection is somewhat obscure, the inverse being referred to the same axes as the

original. The special geometrical form here given to the transformation by means of Dr. Hirst's method of quadric inversion is to be found in the *American Journal of Mathematics*, vol. xiv., p. 301, and vol. xv., p. 221.

244. As regards any specialty of position with respect to OI, the results are very similar. Contact with OI at X, on the line JH, gives a cusp at I, IH being the tangent, etc. Moreover, it is seen from Fig. 53 that contact with OI at I (compartments 1′, 7′, for example) gives contact with OI at I; and contact with OI at O (2′, 13′, for example) gives contact with IJ at I. But if the principal points be not distinct, the results are slightly different; the form they assume will be made plain by the construction of the diagram corresponding to Fig. 53. For instance, if the fundamental conic be a line-pair, a branch cutting OI gives a branch touching IJ at I; hence a branch cutting OI in two points gives a tacnode at I; and a branch touching OI gives a cusp of the second species at I.

Effect of Inversion on a Curve as a Whole.

245. We have here used inversion as a method for analysing singularities; but as it institutes a correspondence between two sets of elements in a plane, it is applicable to the investigation of the properties of a curve as a whole.

As an example, consider the theorem:—If a conic be inscribed in a triangle, the lines joining the points of contact to the opposite vertices are concurrent. Let the triangle be OIJ, and the point of concurrence M. Invert with respect to the conic that touches OI, OJ at I, J, and passes through M. There is now a cusp at O, with OM as tangent, and likewise at I, J; since the original does not pass through O, I, J the inverse does not cut IJ, OI, OJ, except at the points O, I, J already considered. Hence the inverse is a quartic with three cusps, and the cuspidal tangents are concurrent. Arranged in this way, this does not prove that in any quartic with three cusps the cuspidal tangents are concurrent; to prove this, let O, I, J be the cusps, which are certainly not collinear; invert with respect to any conic touching OI, OJ at I, J. Let the cuspidal tangent at O meet IJ at O'; let the tangents at I, J meet the conic of inversion at A, B, and let JA, IB meet OI, OJ at I', J'. Then owing to the cusps at O, I, J the inverse has contact with IJ, OI, OJ at O', I', J'; now the inverse has no other points on the sides of OIJ, it is therefore a conic inscribed in OIJ, and consequently OO', JI', IJ' are concurrent; their inverses,

that is, OO', IA, JB, are therefore concurrent; hence in a tricuspidal quartic the cuspidal tangents are concurrent.

Similarly considering a conic cutting the sides of the triangle in three pairs of real points, the existence of a trinodal quartic is proved, and it is shown that the three pairs of nodal tangents touch a conic. And considering a conic in the various possible positions with regard to the triangle, meeting the three sides in all possible ways, the existence of different varieties of quartics with three double points is made evident.

246. These theorems depend on the correspondence of point to point, and of straight line through a principal point to straight line through a principal point; but to a straight line in general corresponds a conic through OIJ. Consider the general conic and the quartic derived from it by inversion; the tangents to the conic invert into conics touching the quartic and passing through O, I, J; the intersection of any two tangents inverts into the intersection of the tangent conics. Hence there follows the theorem:—Through any point two conics can be drawn to touch a quartic with three double points and pass through the double points.

Ex. 1. Obtain the general equation of a trinodal quartic (§ 101) by inversion.

Ex. 2. Show that three pairs of tangents to a quartic can be drawn from the three nodes; and that these six lines touch a conic.

247. Since the conic and quartic are inverse, the order of a curve may be doubled or halved by inversion; and these are the extreme cases. The application of the formulæ of transformation gives for the inverse an equation whose degree is twice that of the primitive equation; but as all factors that are simply powers of x, y, z are to be rejected (§ 235), the degree of the inverse equation may be lowered. But as inversion applied to this derived equation is to restore the primitive, the degree cannot be lower than one half that of the primitive.

Let the primitive curve of order m have at I, J, O multiple points of orders i, j, k (where any of the numbers i, j, k may be zero); and let it cut OI, OJ, IJ in i', j', k' points, so that

$$i'+i+k=m, \quad j'+j+k=m, \quad k'+i+j=m;$$

that is, $\quad i'=m-i-k, \quad j'=m-j-k, \quad k'=m-i-j.$

The inverse has then at I, J, O multiple points of orders i', j', k', and cuts OI, OJ, IJ in i, j, k points; hence

$$m'=i'+k'+i=j'+k'+j=i'+j'+k;$$

that is,
$$m' = 2m - i - j - k,$$
$$i' = m - i - k,$$
$$j' = m - j - k,$$
$$k' = m - i - j;$$
from which also $m = 2m' - i' - j' - k'$, etc.

Thus for example, a conic through I, J inverts into a conic through I, J; a conic through one principal point, O, inverts into a cubic with a double point at O and ordinary points at I, J. The conic inverts therefore into a cubic with one double point, or a quartic with three double points, each of which is a curve of deficiency zero; and on inverting any curve, it will be found that the deficiency is unaltered by the transformation (compare §§ 143, 288).

248. Since lines through either fundamental point invert into lines through the other fundamental point, it follows that in circular inversion, where the circular points are the fundamental points, isotropic lines invert into isotropic lines; every one inverting into a line through the other circular point, the two of a pair are interchanged, but as they enter only by pairs this does not affect the final result. Hence a focus F inverts into a focus F'. If however O, the centre of the circle of inversion, be itself a focus, OI, OJ being tangents there are cusps at I, J on the inverse, and for this curve O is not necessarily a focus.

Ex. Discuss the inverse of a conic with regard to (1) a focus, (2) the centre, determining all particulars as to order; number, situation, and nature of double points; number and situation of foci.

Reciprocation.

249. We have now to consider the association of the doubly infinite system of points in a plane with the doubly infinite system of lines in a plane. We shall find that all the results obtainable by this were arrived at in the earlier chapters by means of the principle of duality.

In § 73 a hint was given as to a way in which a geometrical connection can be instituted between the points and lines of a plane. Let the polar of P with respect to any proper conic F be denoted by p; thus to the points P, Q, ..., correspond the lines p, q, ...; and since PQ is the polar of pq, to the lines PQ, PR, ..., correspond the points pq, pr,... . Moreover, collinear points P, Q, R, ..., give concurrent lines p, q, r, ..., and the two configurations are homographic; their correspondence is precisely that afforded by

the principle of duality. The two figures thus obtained by means of poles and polars, being reciprocal in their relation, are called Reciprocal Polars; and the process by which one is derived from the other is called Reciprocation; the conic used as a foundation for the process is the auxiliary conic.

If this auxiliary conic be the imaginary conic

$$x^2+y^2+z^2=0,$$

the polar of f, g, h is

$$fx+gy+hz=0,$$

hence to the point f, g, h corresponds the line f, g, h; to the locus of the point f, g, h corresponds the envelope of the line f, g, h; the two curves thus connected by reciprocation with respect to the special auxiliary conic are the curves defined as reciprocal in § 69.

250. The correspondence hitherto studied by means of the principle of duality has affected only non-metric properties. But in reciprocating with respect to a specified conic, all the properties of one figure are derived from those of the other, and consequently it is possible to pass from the metric properties of a figure to those of its reciprocal; but the only case in which this can be done with facility is when the auxiliary conic is a circle. This however does not limit the generality of the method.

The auxiliary conic being a circle, the centre O is the origin of reciprocation. The polar of any point P is constructed by taking on OP a point M, given by $OM.OP=(\text{radius})^2$, and drawing through M a line perpendicular to OP; and similarly the pole of a line is constructed. Hence lines belonging to the primitive that pass through O give points at infinity on the reciprocal curve; for instance, the tangents from O to any curve of the primitive figure and their points of contact with this curve reciprocate into points at infinity on the reciprocal and the tangents at these points; that is, the points of contact of tangents from O to any curve reciprocate into the asymptotes of the reciprocal curve.

251. The theory of reciprocation with respect to a circle is fully treated in Salmon's *Conic Sections*, Chapter XV., and there is therefore no occasion to go into details here. But one special example may be given, for the sake of showing the connection between § 124 and § 134.

A pencil of conics reciprocates into a range; let the origin of reciprocation be taken at a vertex of the self-conjugate triangle, then as the common points P, Q, R, S are in two

pairs on lines through O, the common tangents which determine the reciprocal range are parallel in pairs. If now the pencil be a pencil of circles, that is, a system of coaxal circles, two of the intersections are the circular points; the origin of reciprocation is one of the limiting points of the system. The four intersections P, Q, ω, ω' connect in pairs through O, viz., $P\omega$ and $Q\omega'$ pass through O; the circle passes through ω, and ω therefore reciprocates into the tangent at ω, that is, into $O\omega$, while P reciprocates into another line through ω; similarly ω', Q reciprocate into lines through ω'; let the lines reciprocal to P, Q intersect in F. The reciprocal to the system of coaxal circles when taken with respect to either limiting point is therefore a system of confocal conics having that limiting point as one focus; the other focus is the point F.

Ex. 1. Show that the auxiliary circle can be chosen so that the two limiting points may be the foci of the reciprocal system.

Ex. 2. Discuss the case of a coaxal system with imaginary limiting points.

252. In connection with reciprocal curves it was seen that a cusp with its tangent and an inflexional tangent with its point of contact are reciprocal. This fact is brought out by the upper part of Fig. 58. Four arcs are drawn, meeting at A, and all having the same tangent b; the reciprocal arcs have as their tangent a, the polar of A, and the point of contact is B, the pole of b. An arc and its reciprocal are marked with the same number; thus 1, 3, which make an inflexion at A, make a cusp at B.

The lower part of Fig. 58 shows the reciprocation of a node with a loop. Corresponding to the node N, with the two tangents p, q, there is a double tangent n with the two points of contact P, Q. From O a real tangent OT can be drawn to the loop, this occurs between p and q as we travel round the loop; hence the reciprocal passes through infinity, P and Q being separated by the point at infinity; the asymptote, t, is the reciprocal to T. In § 239 it was shown that a cusp may be regarded as the final form of a node with a loop, when the two tangents close up and the loop disappears. Now when p, q approach coincidence and NT vanishes the points P, Q approach coincidence and also the lines n, t. Hence we have the two arcs that make an inflexional branch, together with all points on the line that is the inflexional tangent; these being reciprocal to the two arcs that make the cuspidal branch, together with all lines through the cusp. Now considering the evanescent loop as an envelope, we see that it does give rise to the two arcs and this assemblage

of lines, and just as this last is not counted as part of the cuspidal branch (envelope), the assemblage of points on the inflexional tangent is not counted as part of the inflexional branch (locus).

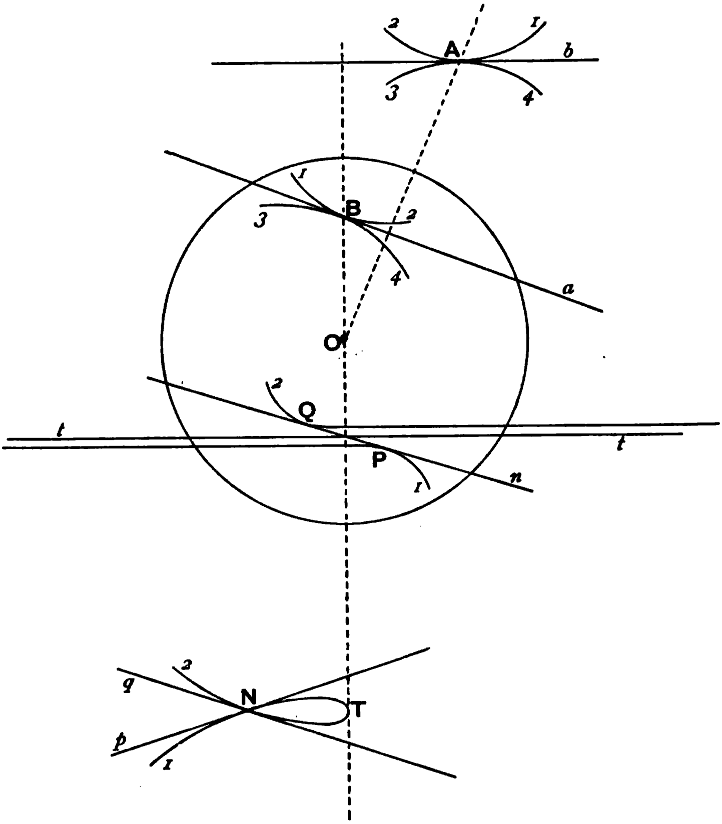

FIG. 58.

The Dualistic Transformation.

253. We have seen that projection presents itself as a special case of linear transformation, specialized however only by position; and that quadric inversion, at first sight a special quadric transformation, differs from the general reversible quadric transformation only by a linear transformation. We have now to show that reciprocation, presenting itself as a special case of the linear dualistic transformation, differs from this only by a linear transformation.

The theory of linear dualistic transformation is sometimes called the theory of skew reciprocation. This name has reference to the fact, now to be proved, that the essentials of the theory are to be found in reciprocation with regard to a fundamental conic; the other name has reference to the underlying essential, the principle of duality, and regards the relation to the auxiliary conic as purely accidental.

Note. This correspondence has also been called Correlation, the figures being correlative (Chasles), but the name is not in general use.

254. In the general dualistic transformation the coordinates of a line are general functions of the coordinates of the corresponding point; the transformation is linear, when these expressions are linear. Hence the formulæ of transformation are

$$\begin{aligned}\xi &= a_1x+a_2y+a_3z,\\ \eta &= b_1x+b_2y+b_3z,\\ \zeta &= c_1x+c_2y+c_3z;\end{aligned}$$

and to a point $P(x, y, z)$ in the first system corresponds a line $p(\xi, \eta, \zeta)$ in the second system; the two systems may be represented in different planes, or the two planes may be superimposed so that we have the two systems in one plane, with it may be different triangles of reference.

Now subject the first system to a linear (point) transformation,

$$\begin{aligned}x' &= a_1x+a_2y+a_3z,\\ y' &= b_1x+b_2y+b_3z,\\ z' &= c_1x+c_2y+c_3z,\end{aligned}$$

by which from a point $P(x, y, z)$ there is derived a point $P'(x', y', z')$. There is now a correspondence between this derived system (P') and the second system (p), expressed by the transformation

$$\xi=x', \quad \eta=y', \quad \zeta=z';$$

these two systems are therefore reciprocal with respect to the auxiliary conic

$$x^2+y^2+z^2=0,$$

which shows that the general linear dualistic transformation differs from the interchange of point and line coordinates only by a collineation.

Note. The term linear transformation is properly used whenever the formulæ of transformation are linear, whether they express point and line coordinates in terms of point and line coordinates or in terms of line and point coordinates; that is, linear transformations include collineations and linear dualistic transformations.

255. Hence so far as the geometrical properties of a figure are concerned, nothing more can be learnt by means of the most general linear dualistic transformation than is recognized intuitively by means of the principle of duality. But considering the correspondence of point and line hereby instituted in its relation to the general theory of (1, 1) correspondence, one or two points require investigation.

Let the two systems be represented in one plane, with the same triangle of reference. To a point x, y, z of the first system corresponds a line ξ, η, ζ of the second system, where

$$\begin{aligned}\xi &= a_1x + a_2y + a_3z,\\ \eta &= b_1x + b_2y + b_3z,\\ \zeta &= c_1x + c_2y + c_3z\,;\end{aligned}$$

equations which may also be written

$$\begin{aligned}x &= \alpha_1\xi + \beta_1\eta + \gamma_1\zeta,\\ y &= \alpha_2\xi + \beta_2\eta + \gamma_2\zeta,\\ z &= \alpha_3\xi + \beta_3\eta + \gamma_3\zeta,\end{aligned}$$

where α_1, α_2, etc. are the minors of a_1, a_2, etc. in the determinant $(a_1b_2c_3)$, and this determinant is rejected as a factor in the expressions for x, y, z.

Hence to a point of the second system,

$$l\xi + m\eta + n\zeta = 0,$$

corresponds the line of the first system

$$l(a_1x + a_2y + a_3z) + m(b_1x + b_2y + b_3z) + n(c_1x + c_2y + c_3z) = 0,$$

that is, to the point l, m, n considered as belonging to the second system there corresponds the line

$$(a_1l + b_1m + c_1n)x + (a_2l + b_2m + c_2n)y + (a_3l + b_3m + c_3n)z = 0.$$

Thus the point x, y, z has two different lines for correspondent when considered as belonging to the two different systems, and the coordinates of these lines are respectively

$$a_1x + a_2y + a_3z, \quad b_1x + b_2y + b_3z, \quad c_1x + c_2y + c_3z\,;$$
$$a_1x + b_1y + c_1z, \quad a_2x + b_2y + c_2z, \quad a_3x + b_3y + c_3z.$$

These two lines coincide if

$$\frac{a_1x + a_2y + a_3z}{a_1x + b_1y + c_1z} = \frac{b_1x + b_2y + b_3z}{a_2x + b_2y + c_2z} = \frac{c_1x + c_2y + c_3z}{a_3x + b_3y + c_3z} = \lambda,$$

where λ satisfies the equation, obtained by eliminating x, y, z,

$$\begin{vmatrix} a_1\lambda - a_1 & b_1\lambda - a_2 & c_1\lambda - a_3 \\ a_2\lambda - b_1 & b_2\lambda - b_2 & c_2\lambda - b_3 \\ a_3\lambda - c_1 & b_3\lambda - c_2 & c_3\lambda - c_3 \end{vmatrix} = 0.$$

This is of the form $\quad \lambda^3 + A\lambda^2 - A\lambda - 1 = 0,$

hence one solution is $\lambda=1$, which gives

$$x:y:z=b_3-c_2:c_1-a_3:a_2-b_1,$$

and there are two others, real or imaginary.

There are therefore three points and their corresponding lines which are associated with one another regardless of the system to which they belong. Thus the general linear dualistic transformation does not definitely associate the points of the plane with the lines of the plane, though this can be done by a special dualistic transformation. The two lines corresponding to a point coincide for every point if the equations

$$\frac{a_1x+a_2y+a_3z}{a_1x+b_1y+c_1z}=\frac{b_1x+b_2y+b_3z}{a_2x+b_2y+c_2z}=\frac{c_1x+c_2y+c_3z}{a_3x+b_3y+c_3z}$$

hold for all values of x, y, z; hence we have the conditions

$$b_3=c_2,\quad c_1=a_3,\quad a_2=b_1,$$

showing that the points of the plane are associated with the lines of the plane only if the equations of transformation assume the form

$$\xi=ax+hy+gz,$$
$$\eta=hx+by+fz,$$
$$\zeta=gx+fy+cz,$$

expressing simply that the point and line are pole and polar with respect to a conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

256. Corresponding elements being different in nature, the question as to elements that coincide with their correspondents (§ 226) is replaced by the question as to elements that are united with their correspondents.

The line ξ, η, ζ is united with the point x, y, z if

$$\xi x+\eta y+\zeta z=0;$$

hence the locus of points that lie on their corresponding lines is the conic

$$(a_1x+a_2y+a_3z)x+(b_1x+b_2y+b_3z)y+(c_1x+c_2y+c_3z)z=0;$$

that is,

$$a_1x^2+b_2y^2+c_3z^2+(b_3+c_2)yz+(c_1+a_3)zx+(a_2+b_1)xy=0,$$

which is called the pole conic. Also the envelope of the lines is a different conic, the polar conic. For the coordinates of a point are given in terms of the coordinates of the corresponding line by the equations

$$x=\alpha_1\xi+\beta_1\eta+\gamma_1\zeta,\ \text{etc.},$$

and therefore the envelope of lines that pass through their correspondents is

$$a_1\xi^2+\beta_2\eta^2+\gamma_3\zeta^2+(\gamma_2+\beta_3)\eta\zeta+(a_3+\gamma_1)\zeta\xi+(\beta_1+a_2)\xi\eta=0,$$

that is,

$$(b_2c_3-b_3c_2)\xi^2+(c_3a_1-c_1a_3)\eta^2+(a_1b_2-a_2b_1)\zeta^2+\eta\zeta(a_3b_1-a_1b_3+c_1a_2-c_2a_1)$$
$$+\zeta\xi(b_1c_2-b_2c_1+a_2b_3-a_3b_2)+\xi\eta(c_2a_3-c_3a_2+b_3c_1-b_1c_3)=0,$$

which is called the polar conic.

We have here been dealing with points of the first system and lines of the second; the equations expressing the correspondence between points of the second system and lines of the first being (§ 255)

$$\xi=a_1x+b_1y+c_1z, \text{ etc.},$$

the pole conic and the polar conic are the same as before.

To compare these two conics, both must be expressed in line coordinates or in point coordinates. Writing the equation of the pole conic in line coordinates (§ 65) the coefficients A, F are proportional to

$$(b_3+c_2)^2-4b_2c_3,$$

that is, $$(b_3-c_2)^2-4(b_2c_3-b_3c_2),$$

and $$2a_1(b_3+c_2)-(c_1+a_3)(a_2+b_1),$$

that is, $$(c_1-a_3)(a_2-b_1)-2(a_3b_1-a_1b_3+c_1a_2-c_2a_1).$$

Hence writing the polar conic in the form

$$\Psi=4(b_2c_3-b_3c_2)\xi^2+\ldots+\ldots=0,$$

the line equation of the pole conic is

$$\Phi=\big((b_3-c_2)\xi+(c_1-a_3)\eta+(a_2-b_1)\zeta\big)^2-\Psi=0.$$

This form shows that the two conics are different, unless the transformation considered is reciprocal; and that they have double contact, the intersection of the common tangents being the point b_3-c_2, c_1-a_3, a_2-b_1, the chord of contact itself being the line $\beta_3-\gamma_2$, γ_1-a_3, $a_2-\beta_1$. Let this line, p, meet the conics in Q, R, and let the intersection of q, r (the tangents at Q, R), be P. Let X be any point on the pole conic, so that the correspondent to X passes through X; by the definition of the polar conic, it is touched by the line corresponding to X; hence the two correspondents to X are the two tangents from X to the polar conic. If therefore X be at Q or at R, the two correspondents coincide; the points Q, R are two of the points determined in § 255, the point P being the third. Hence of the three points con-

sidered two lie on their corresponding lines, and the third does not.

If now the triangle PQR be the triangle of reference, $\xi=0$ corresponds to $x=0$, but $\eta=0$ and $\zeta=0$ correspond respectively to $z=0$ and $y=0$. Hence the formulæ of transformation are

$$\xi=a_1x, \quad \eta=b_3z, \quad \zeta=c_2y.$$

Note. For a fuller discussion of skew reciprocation see Salmon's *Higher Plane Curves*, §§ 332-342; there is a typographical mistake in § 335, where the line equation of the pole conic is given instead of the line equation of the polar conic.

Birational Transformation of a Curve.

257. The transformations hitherto considered are birational transformations of the whole plane; they are Cremona transformations. But there are transformations that are birational only as regards a curve of the plane. For instance, to the locus of a point P there corresponds by reciprocation the envelope of a line p; let P' be the point of contact of p with its envelope, then P' is determined by P, and P by P'. Hence there is a (1, 1) correspondence between the points of the two curves, and also between the lines of the two curves.

For example, the reciprocal to

$$x^3+y^3+z^3=0 \quad \text{.........................(1)},$$

is (§ 68) $$x^6+y^6+z^6-2y^3z^3-2z^3x^3-2x^3y^3=0 \quad \text{..............(2)}.$$

Let x_1, y_1, z_1 be a point on (1); the tangent at this point is

$$x_1^2x+y_1^2y+z_1^2z=0,$$

and to this line corresponds the point

$$x_2:y_2:z_2=x_1^2:y_1^2:z_1^2 \quad \text{....................(3)},$$

a point on (2). The tangent to (2) at this point is

$$(x_2^5-x_2^2y_2^3-x_2^2z_2^3)x+(y_2^5-y_2^2z_2^3-y_2^2x_2^3)y+(z_2^5-z_2^2x_2^3-z_2^2y_2^3)z=0,$$

which can be written

$$x_2^2(2x_2^3-u_2)x+y_2^2(2y_2^3-u_2)y+z_2^2(2z_2^3-u_2)z=0,$$

where u_2 stands for $x_2^3+y_2^3+z_2^3$.

Now x_1, y_1, z_1 corresponds to this line; hence

$$x_1:y_1:z_1=x_2^2(2x_2^3-u_2):y_2^2(2y_2^3-u_2):z_2^2(2z_2^3-u_2)\text{...(4)}.$$

Thus x_2, y_2, z_2 are given rationally in terms of x_1, y_1, z_1 by equations (3), and x_1, y_1, z_1 in terms of x_2, y_2, z_2 by equations (4).

These two sets of equations are equivalent, in virtue of equation (1) or (2); for substituting in (4) the values of x_2, y_2, z_2 given by (3), the result is

$$\begin{aligned} x_1 : y_1 : z_1 &= x_1^4(2x_1^6 - x_1^6 - y_1^6 - z_1^6) : \ldots : \ldots \\ &= x_1^4\left(x_1^6 + 2y_1^3z_1^3 - (y_1^3+z_1^3)^2\right) : \ldots : \ldots \\ &= x_1^4(x_1^6 + 2y_1^3z_1^3 - x_1^6) : \ldots : \ldots \\ &= 2x_1^4y_1^3z_1^3 : 2x_1^3y_1^4z_1^3 : 2x_1^3y_1^3z_1^4 \\ &= x_1 : y_1 : z_1, \end{aligned}$$

that is, an identity.

An algebraic transformation that is birational as regards the points of two curves but not as regards the whole plane is called a Riemann transformation. That is, a Riemann transformation of a curve $F(x, y, z)=0$ is expressed by the equations

$$x : y : z = f_1(x', y', z') : f_2(x', y', z') : f_3(x', y', z'),$$

where f_1, f_2, f_3 are homogeneous polynomials of the same degree k that have no common factor, and are such that (by means of the equation $F=0$) x', y', z' can be obtained in the form

$$x' : y' : z' = \phi_1(x, y, z) : \phi_2(x, y, z) : \phi_3(x, y, z),$$

where ϕ_1, ϕ_2, ϕ_3 are homogeneous polynomials of the same degree κ that have no common factor.

In the example just given the two curves were reciprocal, but this is not necessary.

For example, consider the curves

$$x^3 - x^2z - y^2z = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x^4 - xy^2z - y^2z^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These are unicursal, the coordinates being expressed in terms of a parameter by means of the equations

$$x_1 = \lambda y_1, \quad \lambda^2 x_1 = (\lambda^2+1)z_1 \ldots\ldots\ldots\ldots\ldots\ldots (1)',$$

$$x_2^2 = \lambda y_2 z_2, \quad x_2 = (\lambda^2-1)z_2 \ldots\ldots\ldots\ldots\ldots\ldots (2)'.$$

Associating the points that depend on the same value of λ, we find

$$\frac{x_1}{y_1} = \frac{x_2^2}{y_2 z_2},$$

$$\frac{x_1}{z_1} = 1 + \frac{1}{\lambda^2} = 1 + \frac{z_2}{x_2+z_2} = \frac{x_2+2z_2}{x_2+z_2},$$

therefore
$$x_1 : y_1 : z_1 = 1 : \frac{y_2 z_2}{x_2^2} : \frac{x_2+z_2}{x_2+2z_2},$$

that is, $x_1 : y_1 : z_1 = x_2^2(x_2+2z_2) : y_2z_2(x_2+2z_2) : x_2^2(x_2+z_2) \ldots . (3)$.

Also $$\frac{x_2}{z_2}=\lambda^2-1=\frac{x_1^2-y_1^2}{y_1^2},$$

$$\frac{x_2}{y_2}=\lambda\frac{z_2}{x_2}=\frac{x_1}{y_1}\cdot\frac{y_1^2}{x_1^2-y_1^2}=\frac{x_1y_1}{x_1^2-y_1^2},$$

therefore $$x_2:y_2:z_2=1:\frac{x_1^2-y_1^2}{x_1y_1}:\frac{y_1^2}{x_1^2-y_1^2},$$

that is, $$x_2:y_2:z_2=x_1y_1(x_1^2-y_1^2):(x_1^2-y_1^2)^2:x_1y_1^3\;\ldots\ldots..(4).$$

Thus by the Riemann transformation expressed by equations (3), (4), either of the given curves is transformed into the other.

Ex. Determine (1) the transformation by which the point λ on one of the curves just used is associated with $\frac{1}{\lambda}$ on the other curve; (2) the transformation when the two parameters are connected by a symmetric bilinear relation.

258. There is one other important part of the theory of correspondence of which mention must be made, the correspondence of points on a curve. The simplest example of this is afforded by homographic systems on conics (§§ 192-196); and just as in this case we have to consider points that coincide with their correspondents, so in the more general case we have to consider *united points*, points that coincide with their correspondents. The theory was originated by Chasles (1864), who dealt in his earlier papers with unicursal curves only. The conception applies however to all curves, but it can only be explained in connection with the theory of Higher Plane Curves. For a brief account, reference may be made to Professor Cayley's article, Curve, in the *Encyclopædia Britannica.*

CHAPTER XII.

THE ABSOLUTE.

Résumé of the Argument.

259. We now review briefly the results already obtained, in order to see clearly what ought to be the next step. The guiding principle has been, as stated in § 1, to generalize our conceptions and their expression as far as possible. In accordance with this, we began by examining the idea of coordinates, and were led to the conclusion that the nature and number of the coordinates required depend on the nature of the space and of the element; more precisely, on the number of dimensions of the space, regarded as composed of elements of the assigned nature, or from the other side on the number of degrees of freedom of the element, regarded as moving in the space considered; or again, on the order of the manifoldness, when the elements are regarded as existing in the space. Thus the same coordinates serve for different elements in different spaces, provided that the number of degrees of freedom be the same. Further, the introduction of one more coordinate relieves us from the necessity of attending to actual values; henceforward ratios alone are required.

This examination showed also that the nature of the element is at our disposal; any geometrical entity may be regarded as the element.

Now as the object of our study is plane geometry, which is primarily geometry with the plane as space, the point as element, we are dealing with two-dimensional geometry; and the results obtained are applicable to any other two-dimensional system, for example, to planes through a point. But among all the possible systems of geometry, there is one whose association with that of points in a plane is of special service, viz., the geometry of lines in a plane. This is two-dimensional, we are therefore at liberty to choose it; it relates to plane geometry, and is therefore essentially a part

of our subject. Henceforth our subject is plane geometry, regarded under two absolutely distinct aspects, the two views being however developed simultaneously by a single course of reasoning. Later on we find that a geometrical dependence of the one theory on the other can be introduced; but the two theories are regarded as essentially distinct.

260. The next step was to arrange a system of homogeneous point coordinates with an associated system of line coordinates; and to facilitate the passage from either system to the other certain multipliers that were at our disposal were chosen so that the equation

$$x\xi + y\eta + z\zeta = 0$$

should be the expression of the fact that the point and line were united in position.

Applying these coordinates to systems of lines and curves, geometrical properties were found to be of three kinds. Properties are

(1) Independent of the nature of the coordinates;
(2) Dependent on the nature of the point coordinates;
(3) Dependent on the nature of the line coordinates.

As regards classes (2) and (3), the two views were not developed simultaneously by a single course of reasoning; we were not able to transfer results from the point geometry to the line geometry. Here our generalization was incomplete.

261. The first intimation that the correspondence of the point and line theories was not all-pervading was received when the fundamental identical relation presented itself under different forms in the two theories (§ 20). The coordinates of a point are subject to a linear inequality; but there are exceptional points that escape this condition, these all lie on a certain straight line. The coordinates of a line are subject to a quadratic inequality, but this does not apply to certain exceptional lines, that is, to all lines passing through one or other of two fixed points.

It is precisely this fact—that there *are* points and lines at variance with the fundamental identical relations, viz., one point on every line, two lines through every point—that calls metric relations into existence (§§ 39, 115).

262. We have now to determine why the correspondence between the point and line theories apparently breaks down as regards these exceptional elements.

One step towards the elucidation of this is obtained from

the consideration of the equation of a circle. It was shown in § 117 that the line equation of a circle of infinite radius is degenerate, and represents the circular points. Now the point equation of any circle is

$$x^2+y^2-2(gx+fy)+g^2+f^2-r^2=0,$$

if the angle C be supposed to be a right angle. Let the line infinity be

$$ax+by+cz=0,$$

then the equation of the circle in the homogeneous form is

$$x^2+y^2-2(gx+fy)(ax+by+cz)+(g^2+f^2-r^2)(ax+by+cz)^2=0.$$

Hence the point equation of a circle of infinite radius, obtained by writing in the above $r=\infty$, is

$$(ax+by+cz)^2=0,$$

that is, the line infinity taken twice.

We are therefore led to consider the question of degenerate conics.

Note. The degenerate conics that present themselves most readily to our remembrance are *evanescent* conics, such as $x^2+y^2=0$; but we have just seen that a conic can degenerate by becoming infinite; the question to be considered is therefore the general one of degenerate conics in point or line coordinates.

Degenerate Conics.

263. A degenerate locus of the second order is two loci of the first order, and is therefore a line-pair; and similarly a degenerate envelope of the second class is two envelopes of the first class, that is, a point-pair.

Now a line has not a line equation; hence we cannot expect the line-pair to have a line equation; and yet from one point of view the equation

$$u=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

even when degenerate, has a reciprocal equation, though the process of § 65 fails.

Since $\Delta=0$, the equations

$$\xi=ax_1+hy_1+gz_1,$$
$$\eta=hx_1+by_1+fz_1,$$
$$\zeta=gx_1+fy_1+cz_1,$$

give three equivalent results

$$(bc-f^2)\xi+(fg-ch)\eta+(fh-bg)\zeta=0,$$
$$(gf-ch)\xi+(ca-g^2)\eta+(gh-af)\zeta=0,$$
$$(hf-bg)\xi+(hg-af)\eta+(ab-h^2)\zeta=0;$$

any one of these is the equation of the point of intersection, which therefore presents itself as apparently the reciprocal.

But this is not satisfactory, for the general reciprocal equation is known to be of the second degree. And certainly the linear equation just found will not, on reciprocation, reproduce the original equation of the second degree.

Consider a conic which does not split up into straight lines, though it is approaching this condition. It has a reciprocal

$$A\xi^2+B\eta^2+C\zeta^2+2F\eta\zeta+2G\zeta\xi+2H\xi\eta=0,$$

into which Δ does not enter. Now as the coefficients $a, b, c, \ldots$ change, the values of $A, B, C, \ldots$ will alter, but not the form of this equation. Hence even when $\Delta=0$, we still have the reciprocal

$$(bc-f^2)\xi^2+(ca-g^2)\eta^2+(ab-h^2)\zeta^2$$
$$+2(gh-af)\eta\zeta+2(hf-bg)\zeta\xi+2(fg-ch)\xi\eta=0.$$

But since $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$, we have also $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}=0,$

and therefore this reciprocal equation splits up into factors. Multiplying by $bc-f^2$, the equation can be written

$$\{(bc-f^2)\xi+(fg-ch)\eta+(fh-bg)\zeta\}^2=-\Delta\{c\eta^2-2f\eta\zeta+b\zeta^2\}$$
$$=0, \text{ since } \Delta=0.$$

Hence the reciprocal is the square of

$$(bc-f^2)\xi+(fg-ch)\eta+(fh-bg)\zeta=0,$$

that is, the line equation represents the intersection of the two lines, counted twice.

Similarly the point equation that is reciprocal to a degenerate line equation represents the join of the two points, taken twice.

264. The actual significance of this result is made clearer by considering a simpler form of the equation. The conic

$$\alpha x^2+\beta y^2+\gamma z^2=0$$

has for its reciprocal

$$\frac{\xi^2}{\alpha}+\frac{\eta^2}{\beta}+\frac{\zeta^2}{\gamma}=0.$$

Write $\gamma=-1,\ \alpha=\dfrac{1}{a^2},\ \beta=\dfrac{1}{b^2}$; the equations are now

$$b^2x^2+a^2y^2-a^2b^2z^2=0,$$
$$a^2\xi^2+b^2\eta^2-\zeta^2=0.$$

Assign to a some constant value, and let b vary.

When $y=0$, $$x^2=a^2z^2,$$
that is, calling the point η, B, the lines BP, BP' are independent of b; the points P, P' (Fig. 59) are therefore fixed.

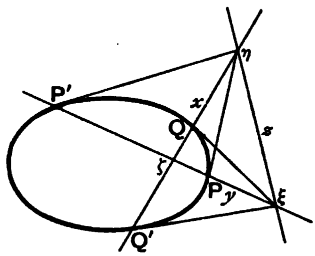

FIG. 59.

Similarly, A being the point ξ, the lines AQ, AQ' are $y^2=b^2z^2$; hence as b diminishes the points Q, Q' approach C (that is, ζ) indefinitely.

Note. In rectangular Cartesians, $z=0$ being the line infinity, PP', QQ' are the axes of the conic; we keep one axis constant, and let the other vanish.

But when $b=0$ the line equation reduces to
$$a^2\xi^2-\zeta^2=0,$$
which represents simply the two points P, P'; and the point equation becomes
$$y^2=0,$$
which represents the indefinite line joining P, P', taken twice.

Thus the degenerate flat conic presents itself as a pair of points; but the only way in which the point equation can express this is by means of the line joining the points, counted twice.

As regards the shape of the flat conic, the Cartesian equation shows that at the points P, Q, the radius of curvature, being respectively $\frac{b^2}{a}$ and $\frac{a^2}{b}$, becomes 0 and ∞.

Note. It must be borne in mind that if we start with the line equation of a pair of points, we get no proper reciprocal at all; and if we start with the point equation of a pair of lines, we get no reciprocal. These reciprocal equations are obtained only by regarding the point-pair or the line-pair as a degenerate conic, that is, as a limiting case.

265. The geometrical significance of this is exhibited in Fig. 60. The conic is an envelope of the second class; as it flattens, changing from (a) to (b) or from (d) to (c), all the

tangents tend to pass through P or P'; and ultimately any line through either of these points belongs to the envelope. Now the conic, quà locus, is the segment PP' (that is, one segment or the other) taken twice; but an ordinary point equation cannot express a terminated portion of a line; it can express only the whole line. Thus the point equation, applicable indifferently to (b) or (c), gives the whole line twice.

Note. In Fig. 60, (a), (b), (c), (d) form a consecutive series, representing the conversion of an ellipse (a) into a hyperbola (d), by the vanishing of one axis, the quantity b^2 changing from positive to negative through zero.

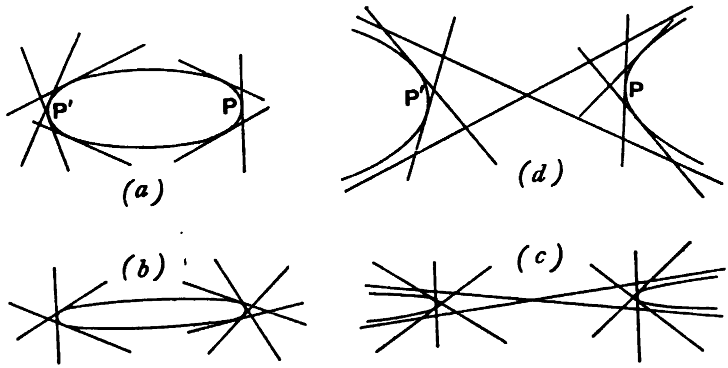

FIG. 60.

The Absolute.

266. Thus it appears that the special points ω, ω', which are fully represented in line coordinates by a degenerate equation of the second degree, cannot be exactly represented in point coordinates; they give simply the line infinity taken twice. The special line that we are led to consider in the use of point coordinates presents itself therefore as a part of the special configuration that we have to consider when using line coordinates; and the circular points are of more fundamental importance than the line infinity. We have seen that the degree of choice allowed in projection enables us to choose ω, ω' arbitrarily, for we can project so that any two points become the circular points (§ 201), but then the figure is determined. Determined, that is to say, as to shape; not as to size, for projection from a plane on to any parallel plane alters the size, but not the shape, and does not affect ω, ω'; not as to position, for ω, ω' are affected neither by translation nor rotation; but absolutely determined as to the metric relations that the parts bear to one

another. Thus the circular points are the absolute elements in the plane, all other elements may be considered as dependent on these.

Now this absolute configuration presenting itself as a *degenerate* conic, the natural generalization is to replace it by a *proper* conic. This proper conic is called *the Absolute*; the notion of the Absolute was introduced by Professor Cayley in his Sixth Memoir upon Quantics, 1859 (*Collected Papers*, vol. ii., No. 158).

If, therefore, we investigate the purely descriptive relations of a system to a general conic, the Absolute, and then make this conic degenerate in the particular way just considered, we shall obtain the relations of the system to the special configuration composed of the circular points and the line infinity taken twice; and interpreting such of these relations as do not prove illusory, we shall obtain the metric properties of the system.

Relation of a Curve to the Absolute.

267. A point at which a curve cuts the Absolute gives a point at infinity on the curve; but the double occurrence of the line infinity in the degenerate Absolute has to be taken into account. For instance, a straight line meets the Absolute in two points; and it is of interest to see what trace is left of this when the Absolute becomes ordinary infinity.

We are dealing with questions involving metric quantities, that is, linear and angular magnitude; and we are examining how far our conceptions of these are in accordance with conclusions drawn from the principle of duality. We must consider therefore in what way linear and angular magnitude correspond.

Imagine a line p to revolve about one extremity O, and let it be indefinite in extent in one direction from O. Let it meet a fixed line in P, and in its initial position a, that is, OA (Fig. 61), let it be perpendicular to this fixed line.

As p revolves about O, so describing angular magnitude, P moves along the line, so describing linear magnitude. When p has described one right angle, P has described the line from A to infinity, on the upper side. As p continues its revolution, describing the second quadrant, P reappears to the left of A; but the line p being terminated at O, we reach the point P' by travelling along the line p, through infinity to P' on the lower side of the line, and accordingly P' describes the line from infinity to A', on the lower side.

Similarly as p describes the third quadrant, the lower side of the line from A' to infinity is described; and as p describes the fourth quadrant, so completing the revolution, P describes the upper side of the line from infinity to A, so completing the description of the double line.

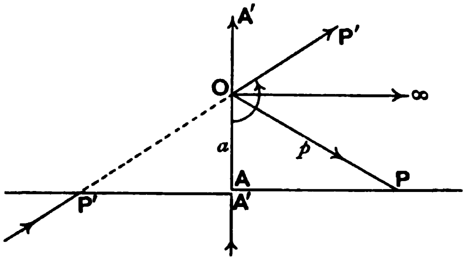

FIG. 61.

Hence in order to exhibit a complete correspondence between linear and angular magnitude, the line must be regarded as double, the upper edge being continuous with the lower edge through infinity.

268. A tangent common to the curve and the Absolute gives rise to an isotropic tangent. If the curve be of class n, there are $2n$ common tangents, falling into n pairs of conjugates. The intersections that do not come at ω, ω' give the foci, of which n are real, $n(n-1)$ imaginary. But if the line infinity be a tangent to the curve, the curve has contact with the Absolute, which is a singularity of position to be considered separately.

269. Contact with the Absolute is represented by contact with the line infinity (Fig. 62, A); or, since every line through either circular point is a tangent at that point, by passage

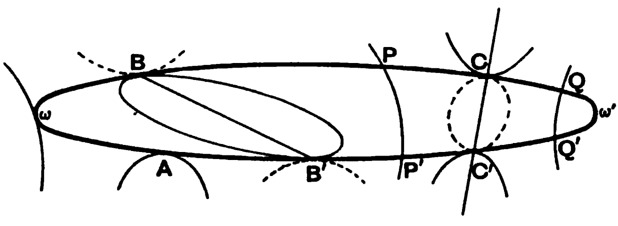

FIG. 62.

through one of the points ω, ω'. But as ω, ω' are conjugate imaginary points, they cannot present themselves separately; there is consequently double contact with the Absolute. For

the case of a conic this is the only way in which double contact can occur; for contact at B, B', where BB' makes with $\omega\omega'$ a vanishing angle, gives a conic that degenerates into the line infinity taken twice; and contact at C, C', where CC' makes with $\omega\omega'$ a finite angle, gives rise to a double point when C, C' come together. Thus a parabola is a conic having contact with the Absolute; a circle is a conic having double contact with the Absolute.

Note. Although the letters ω, ω' appear in Fig. 62, it is important to notice that the *points* ω, ω' have no definite existence until the Absolute is regarded as degenerate.

Correspondence of Asymptotes and Foci.

270. It now appears that the asymptotes of a locus of order n correspond to the foci of an envelope of class n (compare § 129).

For a curve of order n cuts the Absolute in $2n$ points; considering the Absolute as a very flat conic, though not actually degenerate, it is seen that these $2n$ points are in n pairs PP', QQ' (Fig. 62). A line determined by a pair, PP', is ultimately a tangent to the given curve f, and therefore an asymptote; a line determined by points that are not a pair, PQ', or PQ, is ultimately the line infinity. Let the equation of the Absolute be

$$u=0;$$

the curve

$$f-uv=0,$$

where v is any expression of degree $n-2$, is a curve of order n passing through all the intersections of f and u. If therefore v be chosen so that this may split up into n linear factors, it represents one of the sets of n lines determined by the $2n$ points; let v be chosen so that these n lines may be PP', QQ', etc., and let their equations be

$$t_1=0, \quad t_2=0, \quad \ldots, \quad t_n=0;$$

then

$$t_1t_2 \ldots t_n \equiv f-uv.$$

If now the Absolute become ordinary infinity, its point equation, $u=0$, becomes $s^2=0$, where $s=0$ is the line infinity; and the equation $f=0$ becomes

$$f\equiv t_1t_2 \ldots t_n+s^2v_{n-2}=0,$$

which is the ordinary expression for the curve in terms of the asymptotes.

Reciprocally, let the line equation of an envelope of class n be $\phi=0$, and let the line equation of the Absolute be $\theta=0$. Then the curve

$$\phi-\theta\psi=0,$$

where ψ is any expression of degree $n-2$, is a curve of class n touching all the common tangents of ϕ and θ. Choosing ψ so that this may split up into n linear factors, it represents one of the sets of n points determined by the $2n$ lines. Now tangents to the Absolute are ultimately lines through ω, ω', unless they coincide with the line infinity; but this alternative is excluded, for we are not supposing the curve ϕ to have contact with the Absolute. Hence the $2n$ tangents fall into conjugate pairs, giving n real intersections; and we may suppose ψ to be chosen so that the n points given by the linear factors of $\phi-\theta\psi$ are these n real points. Let these factors be $\rho_1, \rho_2, \ldots, \rho_n$; and let the Absolute become ordinary infinity, so that θ becomes $\omega\omega'$; then we have

$$\phi \equiv \rho_1\rho_2 \cdots \rho_n + \omega\omega'\psi_{n-2} = 0$$

as the equation of the curve in terms of the real foci.

Correspondence of Linear and Angular Magnitude.

271. Since the metric properties of a system are descriptive properties of the extended system obtained by combining the Absolute with the given system, it must be possible to give a descriptive definition of linear and angular magnitude; and if there be an exact quantitative correspondence between the two conceptions, we may expect to discover it at this point.

272. It was shown in § 115 that lines, perpendicular according to the ordinary conception, are harmonic with respect to the isotropic lines through their intersection; and accordingly this was adopted as the definition of perpendicularity. In generalizing this, the isotropic lines are replaced by tangents to the Absolute; now lines through a point harmonic with respect to the tangents from that point to a conic, are conjugate with respect to the conic. Hence *lines conjugate with respect to the Absolute are said to be perpendicular.*

To obtain a general definition of the angle between two lines OP, OQ that shall be consistent with the ordinary Cartesian conception, consider what is involved in this idea. Take the line OQ and a perpendicular through O as Cartesian axes of x and y; let OP make with OQ an angle a, then the equation of OP can be written in the form

$$u = x\sin a - y\cos a = 0 \quad\text{.......................(1).}$$

The isotropic lines through O are

$$x \pm iy = 0.$$

Now the given lines OP, OQ are

$$u=0, \quad y=0;$$

expressing the isotropic lines in terms of u and y, by means of equation (1), they become

$$u+y\cos a \pm iy\sin a=0,$$

that is, $$u+y(\cos a \pm i\sin a)=0.$$

The two pairs of lines

$$(u, y), \quad (u+ky, u+k'y)$$

give a pencil whose cross-ratio is $\frac{k}{k'}$; hence the cross-ratio of the pencil determined by OP, OQ and the isotropic lines

$$=\frac{\cos a+i\sin a}{\cos a-i\sin a}=(\cos a+i\sin a)^2=e^{2ia};$$

that is, if the line OP make with the line OQ an angle a, then

$$a=\frac{1}{2i}\log\{O\,.\,PQ,\,\omega\omega'\}.$$

Written in this form, the expression is at once susceptible of generalization, and affords the definition:—

The angle determined by two lines is a constant multiple of the logarithm of the cross-ratio of the pencil formed by the two lines and the two tangents to the Absolute drawn from their intersection.

Calling these two tangents i, j, and writing $\overline{pq}$ for the angle made by p with q, this is

$$\overline{pq}=c\log(pq,\,ij).$$

The value of c is usually taken to be $\frac{1}{2i}$, in order that this generalized measurement of the angle may agree with the

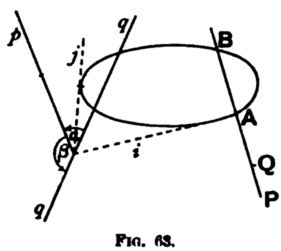

FIG. 63.

ordinary Cartesian measurement when the Absolute becomes ordinary infinity; but for theoretical purposes, c may have any value.

If now the pencil be harmonic,

$$(pq,\,ij)=(qp,\,ij),$$

showing (Fig. 63) that adjacent angles a and β are equal, agreeing with the fundamental property of perpendicular lines.

If the intersection of the lines be on the Absolute, the tangents i and j are now the same line, and consequently

$$(pq,\ ij)=1,$$

that is, $a=\frac{1}{2i}\log(pq,\ ij)=\frac{1}{2i}\log 1=0$; the angle vanishes, agreeing with the ordinary conception of parallelism. Hence for generalized parallel lines we have the definition:—

Lines that meet on the Absolute are parallel;

and it therefore appears that through any point two lines can be drawn parallel to a given line.

273. Reciprocally, we define the generalized distance between two points P, Q by means of the points A, B in which the line PQ meets the Absolute, obtaining

$$\overline{PQ}=c\log(PQ,\ AB)$$

for the distance from P to Q.

When the Absolute becomes ordinary infinity, this agrees with the ordinary measurement of the distance from P to Q, if c be properly chosen. For

$$\begin{aligned}\overline{PQ}&=c\log(PQ,\ AB)\\&=c\log\left(\frac{PA}{QA}:\frac{PB}{QB}\right)\\&=c\log\left(\frac{QA+PQ}{QA}:\frac{QB+PQ}{QB}\right)\\&=c\left\{\log\left(1+\frac{PQ}{QA}\right)-\log\left(1+\frac{PQ}{QB}\right)\right\}\\&=c\left\{\left(\frac{PQ}{QA}-\frac{1}{2}\left(\frac{PQ}{QA}\right)^2+\ldots\right)-\left(\frac{PQ}{QB}-\frac{1}{2}\left(\frac{PQ}{QB}\right)^2+\ldots\right)\right\};\end{aligned}$$

that is, $$\overline{PQ}=c\left\{\frac{AB\,.\,PQ}{QA\,.\,QB}-\frac{(QB^2-QA^2)PQ^2}{2QA^2\,.\,QB^2}+\ldots\right\}.$$

Now the points A, B are at infinity, and it was shown in Chapter II. that all distances QA, QB are absolute constants. Moreover, AB is an absolute constant whose value is indefinitely small. Hence the expression

$$c\,\frac{AB}{QA\,.\,QB}$$

is an absolute constant, and as c is at our disposal, this con-

stant can be made to assume the value unity. The terms following the first term in the expression for PQ vanish compared with the first, hence

$$\overline{PQ} = PQ,$$

that is, the generalized distance becomes the natural distance when the Absolute becomes ordinary infinity.

Note. That AB is an absolute constant is shown by the fact that the Absolute degrades to a repeated line whose direction is indeterminate. Considering, instead of the repeated line, two parallel lines, AB is the distance from one to the other measured along the line AB, and the direction of AB being immaterial, this is a constant.

274. Hence the correspondence of generalized linear and angular magnitude is exact; but in passing to the expression of natural linear and angular magnitude the quantity c, which plays the same part in the two theories in the generalized system, is determined in two different ways. In the line theory, to which angular magnitude belongs, a definite imaginary expression is assigned to c; in the point theory, where linear magnitude is involved, c assumes an indefinite evanescent real value.

275. Since the angle between two lines is defined by their relation to tangents to the Absolute, the case when one of the lines is itself a tangent must be expected to present special features; that this does happen was shown in § 113. The angle that $y = ix$ makes with $y = mx$ is $\tan^{-1} i$, an expression which does not involve m, and is therefore a constant; similarly in the general case now under investigation,

$$\overline{pq} = \frac{1}{2i} \log(pq, ij)$$

when p is i or j, becomes

$$\overline{pq} = \frac{1}{2i} \log 0 \quad \text{or} \quad \frac{1}{2i} \log \infty .$$

Hence the idea of direction cannot properly be associated with the tangents to the Absolute; and in the case of ordinary infinity, the idea of direction must not be associated with the isotropic lines; and this exclusion applies also to the line infinity, which is one of the exceptional lines in the plane.

Note. Writing k for (pq, ij), the expression for the generalized angle is

$$\overline{pq} = \frac{1}{2i} \log k.$$

Now $\log k$ is a many-valued function, with the period $2\pi i$; hence if one value of $\log k$ be $2ai$, the general value is $2ai + 2n\pi i$, from which

$$\overline{pq} = a + n\pi ;$$

that is, given the position of the two lines, the angle that one makes with the other is indeterminate as regards a multiple of π. Similarly the generalized distance from one point to another is indeterminate to the same extent.

Ex. 1. Show that $\overline{PQ}+\overline{QR}=\overline{PR}$; $\overline{pq}+\overline{qr}=\overline{pr}$.

Ex. 2. The distance between two points on an isotropic line is zero.

Ex. 3. Show that inversion with regard to the Absolute increases the distances of all points from the origin by a constant.

Ex. 4. What is the generalized distance from a point to a line?

276. Since the (natural) angle made by two lines is defined by means of the pencil formed by the two lines and the isotropic lines through their intersection, the angle made by the asymptotes of a conic can be expressed in terms of the points in which the conic cuts the line infinity. If these points be P, P', then for the angle between the asymptotes we have the expression

$$\frac{1}{2i}\log(PP',\,\omega\omega').$$

Conics for which this expression has the same value, that is, conics for which the cross-ratio $(PP',\,\omega\omega')$ has the same value, are said to be similar; and conics for which the points P, P' are the same are similar and similarly placed; such conics are called homothetic.

The Generalized Normal and Evolute.

277. The generalized normal to a curve at a point P is the line through P, conjugate to the tangent at P, and is therefore at once constructed by joining P to the pole of the tangent at P. That is, the generalized normal is the line joining corresponding points on the curve and its reciprocal with regard to the Absolute. Hence the curve and its reciprocal with regard to the Absolute have the same normals. The envelope of these normals is the generalized evolute; this is therefore an envelope symmetrically derived from the curve and its reciprocal with regard to the Absolute. But the Absolute may be any conic; thus connected with a curve and its reciprocal with regard to any auxiliary conic we may consider a special envelope, the envelope of the line joining corresponding points, and a special locus, the locus of the intersection of corresponding tangents. But when the auxiliary conic is allowed to degenerate into a pair of points, this locus has no significance. The whole question however belongs properly to the theory of Higher Plane Curves.

General Considerations.

278. We now consider what we have gained by introducing the conception of the Absolute. *We have completed the generalization whose incompleteness was pointed out in* § 260. We have obliterated the distinction between descriptive properties and metric properties, showing that these last are descriptive properties of the extended system composed of the original system and the Absolute. *The whole of our geometry is therefore projective;* and also the principle of duality is applicable throughout.

But one important point must be noticed. *The system of geometry developed in these chapters does not comprise the whole of two-dimensional geometry.* There is a two-dimensional geometry in which the conception of a line at infinity finds no place; in this system infinity is a point. Now the investigation in Chapter II. which led us to the conclusion that the exceptional point elements are arranged on a straight line, lying entirely at infinity, left no margin of choice, and involved no assumptions beyond any that were made in Chapter I. We had to account for the vanishing of $a\alpha+b\beta+c\gamma$, that was, we had to account for a certain straight line. The argument in §§ 8, 9 depended on the assumptions that two straight lines meet in one point only, that is, that two straight lines determine a point uniquely, and that two points determine a straight line uniquely. If then there be any system of two-dimensional geometry in which two primary elements do not determine the secondary element uniquely, or in which two secondary elements do not determine a primary element uniquely, then the processes and results of these investigations cannot be directly applied to this system.

279. That there is a two-dimensional geometry of this nature is easily seen. Let a sphere touch a plane at O, and let O' be the other extremity of the diameter through O. Let P be any point in the plane, and let $O'P$ meet the sphere in P'. Thus any figure in the plane is represented point for point by a figure on the sphere, in which, in general, a curve corresponds to a curve. In particular, a circle on the plane projects into a circle on the sphere, and a circle on the sphere projects into a circle on the plane. But a circle passing through O' projects into a straight line in the plane, and a straight line PQ in the plane projects, by means of the plane $O'PQ$ (Fig. 64), into a circle $O'P'Q'$ on the sphere, having its tangent at O' parallel to PQ; the point at infinity on the line PQ projects into the point O', the projecting line being

this tangent.* Hence the projections of all straight lines have this point O' in common; and considering any two straight lines that intersect in P, their projections on the sphere have in common the two points P', O'. If however the two lines considered be parallel, P' approaches O' indefinitely.

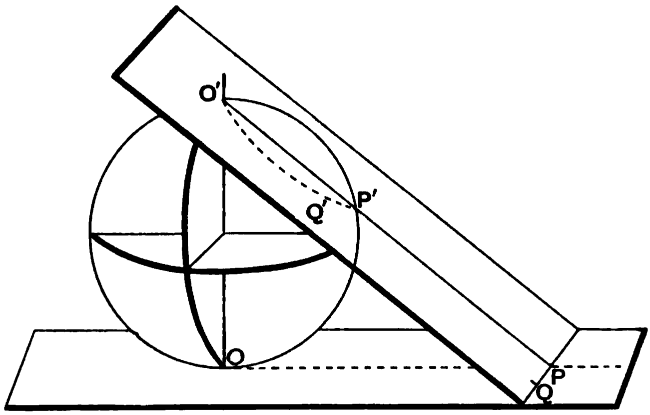

FIG. 64.

Now imagine the radius of the sphere to increase indefinitely, so that for any finite distance from O the spherical surface cannot be distinguished from the plane. The geometry on the sphere is therefore the same as the geometry on the plane over the whole finite region; but infinity is now a point instead of a straight line. The primary element is still the point; but the secondary element is a circle, specialized however by passing through the point infinity,

* Taking O' as origin, any circle in the plane through O has for its equations

$$z=2a, \quad x^2+y^2+lx+my+n=0 \quad\text{.........................(1).}$$

Hence the projecting conical surface is

$$x^2+y^2+(lx+my)\frac{z}{2a}+n\frac{z^2}{4a^2}=0 \quad\text{.........................(2),}$$

and this cuts the spherical surface

$$x^2+y^2+z^2-2az=0 \quad\text{.........................(3),}$$

in the required projection. Subtracting (3) from (2),

$$(lx+my)\frac{z}{2a}+(n-4a^2)\frac{z^2}{4a^2}+2az=0$$

is the equation of a surface through the intersections of the cone and the sphere; but this is the product of two linear factors, and it therefore represents the planes

$$z=0, \quad 2a(lx+my)+(n-4a^2)z+8a^3=0.$$

Thus the curve of intersection is the intersection of the sphere and a plane; hence it is a circle.

and throughout the finite region indistinguishable from a straight line. Calling this specialized circle a straight line, two straight lines are now to be considered as having two points of intersection, of which one may be at a finite distance. Hence the three sides of the triangle of reference have a common point, the point O'; this point lies therefore on the lines

$$\alpha=0, \quad \beta=0, \quad \gamma=0,$$

that is, O' is the point 0, 0, 0; and the paradoxical equation

$$a\alpha+b\beta+c\gamma=0$$

is to be interpreted as referring simply to this point.

Thus it is seen that projective geometry of two dimensions does not include *all* two-dimensional geometry; we have here described one two-dimensional geometry that is not projective, and there are other systems.

280. This spherical geometry agrees with projective geometry in requiring two independent coordinates; the results obtained are identical over the whole finite region; but it does not follow that it is most conveniently treated by this same method of homogeneous coordinates. The proper system of coordinates requires independent investigation, which cannot be undertaken here*; but one general remark may be made, viz., that the general discussions of two-dimensional geometry contained in the preceding pages are applicable to this spherical geometry at least so far as the argument is conducted in terms of the elements, and not of the coordinates.

281. The generalized idea of an angle is due to Laguerre; the generalized idea of distance to Professor Cayley, who originally used the inverse cosine, but then adopted Professor Klein's modification, by which the distance is expressed as a logarithm.

For more detailed discussion of the theory of the Absolute, reference should be made to Professor Cayley's paper (see § 266) in the notes to which other references will be found; of later date than is covered by these references there is another paper by Professor Klein in the *Mathematische Annalen*, t. xxxvii. See also Sir Robert Ball's article on Measurement, in the *Encyclopædia Britannica*; Clifford's paper on Analytical Metrics (*Mathematical Papers*, p. 80; 1864); and Professor Klein's lectures on *Nicht-Euklidische Geometrie*, t. i., pp. 61, etc.

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* The development of this system of geometry ("the geometry of reciprocal radii," Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, § 6) is to be found in writings in which the complex variable is employed.

CHAPTER XIII.

INVARIANTS AND COVARIANTS.

Groups of Transformations.

282. We have now realized that the whole of the geometry under consideration is projective. In §§ 207, 211, it was shown that projective properties are those that are unaltered when the figure is subjected to a linear transformation; hence in considering the abstract laws of being of the geometrical configurations in question, putting aside as accidental and merely illustrative (§ 229) their manifestation by means of these geometrical configurations, the important fact is that the expression of any property is unaltered by linear transformation.

Note. It was shown that if from an equation $F=0$ we derive a new equation $F'=0$ by a linear transformation, then (i.) the two curves $F=0$ and $F'=0$ can be placed in perspective; (ii.) the two equations $F=0$, $F'=0$ can be regarded as two equations of the same curve with different triangles of reference.

The former is the natural view when the subject is considered geometrically; the latter is the natural algebraic view. In the former the projective properties of the two figures are the same; in the latter the two expressions of any projective property of the one figure are the same. For example, let $F=0$ be a degenerate point equation of the second degree. Then by the first interpretation, F is a line-pair, and therefore also F' is a line-pair. By the second interpretation, if F be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$

the coefficients are connected by a relation

$$\Delta=abc+2fgh-af^2-bg^2-ch^2=0\text{ ....(1)};$$

and as the transformed expression F', that is,

$$a'x'^2+b'y'^2+c'z'^2+2f'y'z'+2g'z'x'+2h'x'y',$$

is also a product of linear factors (for it is simply the same expression, written differently), the new coefficients are connected by a relation

$$\Delta'=a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2=0\text{ ....(2)}.$$

Here (1) and (2) are the two expressions of a projective property of the one figure; and that these two are the same appears from § 289 where it is shown that

$$\Delta'=D^2\Delta,\text{where } D\neq 0,$$

hence of the two equations (1), (2), each implies the other.

283. In this abstract consideration of the subject, the conception of a group of operations (in this case, transformations) presents itself.* A system of operations is a group when the operation resulting from the combination of any number of the contained operations is itself a member of the system. If a body be moved from any position A to another position B, and then from B to C, the total effect is simply that it is moved from A to C; the combination of two movements makes a movement, and nothing else; all movements form a group. Dividing movements into translations and rotations (and thus not considering a translation as rotation about a point at infinity), the translations form a group by themselves, for a translation combined with a translation gives a translation; but the rotations do not form a group, since two rotations may result in a translation. The group of translations extended by the rotations, these not forming a group, forms the whole group of movements; and the whole group of movements contains certain smaller groups, as for example the group of translations, and the group of rotations about any arbitrarily chosen point.

284. Confining ourselves now to the plane, the group of translations and rotations leaves unaltered the circular points, and therefore also the line infinity as a whole, but not the separate points on it. The sub-group of translations leaves unaltered every point on the line infinity; the sub-group of rotations about a fixed point leaves unaltered the circular points and the fixed point. Thus the sub-groups contained in a given group are differentiated by leaving unaltered some configuration that is not left unaltered by all members of the given group.

Two collineations produce a collineation; hence collineations form a group. This includes the group of movements, which is characterized by leaving the circular points unaltered; and as before, other sub-groups may be selected by the property of leaving unaltered an arbitrary configuration, for it is plain that if transformations A and B leave certain elements unchanged, then the transformation resulting from the two leaves these elements unchanged.

The combination of two dualistic transformations is not a dualistic transformation; it is a collineation. Hence the dualistic transformations do not form a group by themselves; but as the combination of a dualistic transformation and a collineation is a dualistic transformation, we may extend the

* See §§ 1, 2 of Professor Klein's *Vergleichende Betrachtungen*, already referred to in § 229.

group of collineations by adding to it all the linear dualistic transformations; we thus obtain the group of linear transformations.

285. Since different points and lines are left unchanged by different linear transformations, none are unchanged by the complete group. If we wish a curve to be unchanged as a whole (considering at present only the points, not the lines), let the original equation be

$$F = (a_1, a_2, \ldots \between x, y, z)^k = 0,$$

then if the formulæ of transformation be

$$x = l_1x' + m_1y' + n_1z', \text{ etc.}$$

the transformed equation is

$$F' = (a_1, a_2, \ldots \between l_1x' + m_1y' + n_1z', \quad , \quad)^k = 0,$$

that is, $$F' = (a_1', a_2', \ldots \between x', y', z')^k = 0,$$

where a_1', a_2', etc. involve the coefficients of transformation. Since the form of the equation is to be unaltered by the transformation, these coefficients l_1, m_1, n_1, l_2, etc. must satisfy the equations

$$a_1' : a_1 = a_2' : a_2 = \ldots \text{etc.} \qquad \text{(1)}.$$

We have therefore 8 quantities wherewith to satisfy equations (1); thus we have more than sufficient if the desired stationary curve be a straight line or a conic, the number of equations in these two cases being 2 and 5 respectively. The number of equations for a stationary cubic is 9; but as we have no assurance that these are independent (for every cubic), we cannot say without further examination that they cannot be satisfied; this question we leave.

Returning to the conic, since there are 8 quantities, and only 5 equations, the number of solutions is certainly triply infinite; and if the equations be not independent, it will be greater. But the equations *are* independent; for if they be not independent in the general case, they cannot be independent in a special case. Now using rectangular Cartesian coordinates, the circular points are unaltered by any transformation; but here are three constants involved, viz., the coordinates of the new origin, and the angle through which the axes are turned; hence in this special case a three-fold infinity of transformations leaves a special conic unaltered, and the equations are seen to be independent. Thus in general the transformations that leave a given conic unaltered form a three-fold infinity.

Ex. 1. Show that the transformations that leave unaltered a conic and a straight line are singly infinite in number; and that a triangle is unaltered by a two-fold infinity of transformations.

Ex. 2. By means of a certain linear point transformation the points on a conic become other points on the conic; show that by the associated line transformation (§ 34) tangents to the conic become other tangents to that same conic.

Ex. 3. By a certain linear dualistic transformation the points on a conic become tangents to that conic; show that the tangents to the conic become points on the conic.

286. The considerations here presented show that the ordinary orthogonal transformations (§ 216) form the included three-fold group that leaves the Absolute unchanged. Now in considering the metric properties of a system of curves directly, we confine ourselves to rectangular Cartesians; the properties are unchanged by this group. But we have seen that instead of confining ourselves to the direct investigation of metric properties, we may use projective methods, and investigate the relation of the system to the Absolute. Dealing with projective properties, the group is now the more extensive group of linear transformations. Thus we may extend the group, if at the same time we adjoin the Absolute.

We now pass on to consider the group of linear transformations, these including collineations and dualistic transformations; and as regards any system of curves, we have to investigate

(i.) the properties of the system by itself;

(ii.) the properties of the system extended by the adjunction of the Absolute.

Linear Transformations.

287. The question of linear transformation may be considered purely algebraically, without any reference to possible geometrical interpretations. When a system of equations in any number of variables is subjected to a linear transformation, certain related expressions are unaltered; a complete knowledge of all the unalterable expressions of the system is tantamount to the knowledge of everything essential in the system. The subject is treated under this aspect in works on Modern Higher Algebra; the Theory of Binary Forms, the Theory of Ternary Forms, the Theory of Invariants, are a few of the special titles given to such works. It is here considered on account of its geometrical significance, and only in such detail as is necessary to show its connection with the foregoing chapters. Hence, though the general algebraic theory applies to homogeneous expressions in any number of variables, the only parts to be taken into account are those relating to binary and ternary quantics,

these finding their geometrical interpretation in the one and two-dimensional geometries that have been considered.

288. When a curve is subjected to any transformation, we consider naturally which of its properties are altered, and which remain. If the transformation be linear, order, class, number and nature of singular points and lines, all remain unaltered; if the transformation be not linear, these will be altered; for instance, if the curve be inverted with regard to a conic, order and class are altered, multiple points are gained and lost, inflexional and double tangents are gained and lost (§§ 237, 247). Thus these numbers associated with the original curve are altered. But it will be found in every case that the number expressing the deficiency is unaltered by quadric inversion; a conic, which is a curve of deficiency 0, inverts into a conic, a cubic with one double point, or a quartic with three double points (§ 247); that is, into a curve of deficiency 0. It is found that *the deficiency is unaltered by any birational transformation*; that is, any two curves that have a (1, 1) correspondence have the same deficiency (§ 143); but for the proof of this theorem we must refer to works on Higher Plane Curves and the Theory of Functions.

If from any system of expressions there be derived an expression, numerical or literal, endowed with the property of remaining invariable when the system is subjected to the transformations of a group, this expression may be called an *invariant* of the system for the group; the problem is therefore (Klein, loc. cit.) to develop for the system the theory of invariants relating to the group of transformations.

Thus the numbers expressing order, class, etc., being invariable under linear transformations might be called invariants; but the name in this case is not so used; it has a specialized meaning and is used only as referring to a special class of derived algebraic expressions. *An invariant of a system is a function of the coefficients whose vanishing expresses a projective property of the system.*

289. The way in which invariants present themselves may be shown by the example already referred to in § 282. Let the expression

$$F=ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

be subjected to a linear transformation,

$$\begin{aligned} x&=l_1x'+m_1y'+n_1z',\\ y&=l_2x'+m_2y'+n_2z',\\ z&=l_3x'+m_3y'+n_3z', \end{aligned}$$

by which F becomes F'. Form the expression Δ from the coefficients of F, and similarly Δ' from the coefficients of F'; then will Δ' contain Δ as a factor. For

$$\begin{aligned}
a' &= al_1^2 + bl_2^2 + cl_3^2 + 2fl_2l_3 + 2gl_3l_1 + 2hl_1l_2,\\
b' &= am_1^2 + \dots\dots,\\
c' &= an_1^2 + \dots\dots;\\
f' &= am_1n_1 + bm_2n_2 + cm_3n_3 + f(m_2n_3 + m_3n_2)\\
&\qquad + g(m_3n_1 + m_1n_3) + h(m_1n_2 + m_2n_1),\\
g' &= an_1l_1 + \dots\dots,\\
h' &= al_1m_1 + \dots\dots;
\end{aligned}$$

hence writing

$$\begin{aligned}
\lambda_1 &= al_1 + hl_2 + gl_3,\\
\lambda_2 &= hl_1 + bl_2 + fl_3,\\
\lambda_3 &= gl_1 + fl_2 + cl_3, \text{ etc.,}
\end{aligned}$$

the expressions for the transformed coefficients become

$$\begin{aligned}
a' &= l_1\lambda_1 + l_2\lambda_2 + l_3\lambda_3,\\
b' &= m_1\mu_1 + m_2\mu_2 + m_3\mu_3,\\
c' &= n_1\nu_1 + n_2\nu_2 + n_3\nu_3;\\
f' &= m_1\nu_1 + m_2\nu_2 + m_3\nu_3 = n_1\mu_1 + n_2\mu_2 + n_3\mu_3,\\
g' &= n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 = l_1\nu_1 + l_2\nu_2 + l_3\nu_3,\\
h' &= l_1\mu_1 + l_2\mu_2 + l_3\mu_3 = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3.
\end{aligned}$$

The expression Δ', written as a determinant, is therefore, by means of the two forms for f' etc.,

$$\begin{vmatrix}
l_1\lambda_1 + l_2\lambda_2 + l_3\lambda_3 & l_1\mu_1 + l_2\mu_2 + l_3\mu_3 & l_1\nu_1 + l_2\nu_2 + l_3\nu_3\\
m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 & m_1\mu_1 + m_2\mu_2 + m_3\mu_3 & m_1\nu_1 + m_2\nu_2 + m_3\nu_3\\
n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 & n_1\mu_1 + n_2\mu_2 + n_3\mu_3 & n_1\nu_1 + n_2\nu_2 + n_3\nu_3
\end{vmatrix},$$

which is $$\begin{vmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3 \end{vmatrix} \times \begin{vmatrix} \lambda_1 & \mu_1 & \nu_1\\ \lambda_2 & \mu_2 & \nu_2\\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix}.$$

The second determinant in this product is itself the product of

$$\begin{vmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3 \end{vmatrix} \text{ and } \begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c \end{vmatrix};$$

hence writing D for the determinant $(l_1m_2n_3)$,

$$\Delta' = D^2\Delta.$$

290. This determinant D, formed with the coefficients of transformation, is the modulus of transformation (§ 34). It

is important to notice that in no circumstances can D vanish; for the vanishing of D implies the concurrence of the lines x, y, z. Moreover, the non-evanescence of D is the only limitation in the choice of values for $l_1, m_1, \ldots$; hence D can be made to have any assigned value other than zero; and any other function of $l_1, m_1, \ldots$, *can* be made to vanish by properly choosing $l_1, m_1, \ldots$; D is therefore the only expression in $l_1, m_1, \ldots$, that does not vanish for some transformation.

291. Let the equations of a system be $S_1=0$, $S_2=0$, etc., and let the various sets of coefficients involved be $a_1, b_1, \ldots, a_2, b_2, \ldots$. Let any S become S' by linear transformation, that is, let the substitution of

$$l_1x'+m_1y'+\ldots \text{ for } x,$$
$$l_2x'+m_2y'+\ldots \text{ for } y, \text{ etc.},$$

change $(a, b, \ldots \between x, y, \ldots)^n$ into $(a', b', \ldots \between x', y', \ldots)^n$;

hence

$$(a', b', \ldots \between x', y', \ldots)^n \equiv (a, b, \ldots \between l_1x'+m_1y'+\ldots, l_2x'+m_2y'+\ldots, \ldots)^n;$$

comparing coefficients,

$$a'=(a, b, \ldots)^1(l_1, m_1, \ldots, l_2, m_2, \ldots)^n, \text{ etc.},$$

that is, the new coefficients in an equation of degree n are linear in the old coefficients and of degree n in the coefficients of transformation.

Since we are concerned only with the ratios of the coefficients in any one of the expressions S, every expression that we have to deal with is homogeneous in every set of coefficients separately. Let G be a function of the coefficients of the system, whose vanishing expresses that the system has some projective property, and let G' be the same function of the new coefficients; let the degree in the several sets of coefficients be $p_1, p_2, \ldots$. Then G' can be expressed in terms of the old coefficients and the coefficients of transformation. *Thus expressed, G' is G multiplied by a power of the modulus of transformation.* For by hypothesis the vanishing of G entails the vanishing of G'; hence

$$G'=MG \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Consider any one set of coefficients, $a, b, \ldots$; the expressions for $a', b', \ldots$ in terms of $a, b, \ldots$ show that the degree in which $a, b, \ldots$ are found in G' is the same as the degree in which they are found in G; hence M does not contain $a, b, \ldots$, and therefore M is a function simply of the coefficients of transformation, homogeneous and of degree $\Sigma n_1 p_1$. Now the vanishing of G' is to imply the vanishing of G, and nothing

else; hence M cannot vanish for any possible transformation, and consequently, by § 290, M can only be a numerical multiple of a power of D. The number of variables being k, the determinant D is of order k, that is, D is of degree k in the coefficients of transformation; hence M is a numerical multiple of $D^{\frac{\Sigma n_1 p_1}{k}}$; that is,

$$G' = KD^{\frac{\Sigma n_1 p_1}{k}} G \qquad (2).$$

To determine the numerical multiplier K, consider the identical transformation $x = x'$, $y = y'$, ... for which $D = 1$; this shows that $K = 1$; and consequently

$$G' = D^{\frac{\Sigma n_1 p_1}{k}} G \qquad (3).$$

This equation is the algebraic expression of invariance; the algebraic definition of an invariant is the following:—

Any function of the coefficients that is unchanged by linear transformation, save as to a power of the modulus of transformation, is called an Invariant.

292. If we have two invariants I, J, belonging to the same system, let

$$I' = D^f I, \qquad J' = D^g J,$$

and let $f = ih$, $g = jh$, where h is the greatest common measure of f, g. Then

$$I'^j = D^{ijh} I^j, \quad J'^i = D^{ijh} J^i,$$

hence $\frac{I'^j}{J'^i} = \frac{I^j}{J^i}$, and writing G for $\frac{I^j}{J^i}$, this shows that $G' = G$; that is, the function G is absolutely unalterable by linear transformation; G is an *absolute invariant*. Plainly an absolute invariant is of zero dimensions, for otherwise it would be affected by the transformation

$$x = lx', \quad y = ly', \; \ldots.$$

From the two invariants I, J, another can be derived. For if

$$H = I^j + J^i,$$

then $$H' = I'^j + J'^i = D^{ijh} I^j + D^{ijh} J^i = D^{ijh} H,$$

and therefore H is an invariant. But being expressible rationally in terms of I and J, H is not counted as a distinct invariant.

293. Given a system of curves, there may be a certain locus projectively related to the system. For instance, we have seen that the locus of a point harmonically subtended

by two conics is a conic; and as the harmonic property is projective, the relation of this conic to the given conics is unaltered by projection. Let the given conics be $S_1=0$, $S_2=0$, and let this derived conic be $V=0$; let S_1, S_2 by linear transformation become S_1', S_2', and let V' be derived from S_1', S_2' exactly as V from S_1, S_2, then V' is what V becomes by linear transformation; for V', the locus derived from S_1', S_2', is the projection of V, the locus derived from S_1, S_2. *A locus, thus projectively connected with a system of loci*, and reciprocally, *an envelope projectively connected with a system of envelopes, is a covariant of the system.* Hence a covariant involves the variables as well as the coefficients. Let the order of the covariant (that is, the degree in the variables) be q, and let the degree in the several sets of coefficients be $p_1, p_2, \ldots$, which may be symbolically expressed by writing

$$V=(a_1)^{p_1}(a_2)^{p_2}\,.\,.\,(x, y, \ldots)^q;$$

then
$$V'=(a_1')^{p_1}(a_2')^{p_2}\ldots(x, y, \ldots)^q.$$

But by the linear transformation in question, V is to become V', save as to a factor whose one characteristic is that it cannot vanish. Applying to V the linear transformation, it becomes

$$V_1=(a_1)^{p_1}(a_2)^{p_2}\ldots(l_1x'+m_1y'+\ldots,\ l_2x'+m_2y'+\ldots,\ \ldots)^q,$$

and we are to have
$$V'=MV_1\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

when V' and V_1 are expressed in terms of the same quantities. Expressing in terms of $a_1, b_1, \ldots, x', y', \ldots$, a comparison of degrees on the two sides of this equation shows that M contains only the coefficients of transformation; let the degree of M in these coefficients be h; the degree on the right hand side is therefore $h+q$, and on the left it is Σn_1p_1; hence

$$h=\Sigma n_1p_1-q.$$

Since M is a homogeneous function of the coefficients of transformation which cannot vanish for any possible transformation, it is a numerical multiple of a power of D; and as the degree of D in $l_1, m_1, \ldots$ is k,

$$M=KD^{\frac{h}{k}},$$

and therefore
$$V'=KD^{\frac{h}{k}}V_1\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The suffix in V_1 simply indicates that $x, y, \ldots$ occurring in V are expressed in terms of $x', y', \ldots$; it may therefore be discarded. The value of K, found as in § 291, is unity; and

the fact that V is a covariant of degrees $p_1, p_2, \ldots$ and of order q, is expressed by the equation

$$V' = D^{\frac{1}{k}(\Sigma n_1 p_1 - q)} V \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Though the idea of a covariant is here introduced by means of curves, that is, with reference to ternary quantics, the argument does not require that the number of variables be specified; we have arrived at the general algebraic idea of a covariant, which is formulated in the definition :—

Any function of the coefficients and variables that is unchanged by linear transformation, save as to a power of the modulus of transformation, is called a Covariant.

From the meaning of invariants and covariants it is at once evident that an invariant or covariant of a covariant is an invariant or covariant of the original system.

Binary Quantics.

294. One geometrical interpretation of the binary quantic being by points on a line, the n-ic represents n points. If two of these points coincide, they coincide in any projection; the condition for the coincidence of two points, being simply the condition for equal roots in the non-homogeneous equation $f(x)=0$, is obtained by the elimination of x from the two equations

$$f(x)=0, \quad \frac{\partial f}{\partial x}=0,$$

or, transferring to the homogeneous form (compare § 99), by the elimination of x, y from the two homogeneous equations

$$\frac{\partial f}{\partial x}=0, \quad \frac{\partial f}{\partial y}=0.$$

The expression thus found, whose vanishing expresses the projective property of coincidence, is called the discriminant of the quantic; and we see that the discriminant is an invariant.

If $n=2$ this is the only invariant; the only projective relation possible for two points is that of coincidence. Writing the quantic in the form $(a, b, c \between x, y)^2$, or in the non-homogeneous form,

$$(a, b, c \between x, 1)^2, \text{ that is, } ax^2+2bx+c,$$

the discriminant is $ac-b^2$; and it is at once seen that

$$a'c'-b'^2=(l_1m_2-l_2m_1)^2(ac-b^2),$$

so agreeing with equation (3) of § 291. There is not any covariant, for there are no definite points specially associated with a pair of points.

295. If $n=3$, there is again only the one invariant, viz.,

$$\Delta = a^2d^2 + 4ac^3 - 6abcd + 4b^3d - 3b^2c^2;$$

and in accordance with the formula,

$$\Delta' = D^6\Delta.$$

There are in this case two covariants; for taking the three points given by the cubic

$$ax^3 + 3bx^2 + 3cx + d = 0$$

as the 1, 2, 3 of § 179, and completing the harmonic range in the three possible ways, we obtain a second triad of points 1′, 2′, 3′; hence there is a covariant of order 3. Also § 179 proves the existence of a pair of points projectively related to the cubic, either of the pair serving to complete the equi-anharmonic range; hence there is a covariant of order 2. Let x' be one of the triad, that is, let the points given by the quartic

$$(x-x')(ax^3 + 3bx^2 + 3cx + d) = 0$$

be harmonic. The condition for this was found in § 155, (i.); applying that condition,

$$g_3 = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3 = 0,$$

to the present case it is found that x' must satisfy the cubic equation

$$(a^2d - 3abc + 2b^3,\ abd + b^2c - 2ac^2,\ 2b^2d - acd - bc^2,$$
$$3bcd - ad^2 - 2c^3 \between x, 1)^3 = 0.$$

Similarly we obtain the covariant of order 2 by applying the condition

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$$

(§ 155, (ii.)) to the quartic

$$(x-x')(ax^3 + 3bx^2 + 3cx + d) = 0,$$

obtaining the result that x' must satisfy the equation

$$(ac - b^2,\ ad - bc,\ bd - c^2 \between x,\ 1)^2 = 0,$$

where the arrow-head on the bracket indicates that the equation is to be written without binomial coefficients.

Since an invariant or covariant of a covariant is an invariant or covariant of the original quantic, we consider what we obtain from the cubicovariant and quadricovariant. As regards the latter it is at once evident that the only invariant, the discriminant, is simply the discriminant of the original cubic; and that this must be so is evident à priori; for coincidence of the points x_1, x_2 (§ 179) can only be due to a coincidence among 1, 2, 3. But also nothing

new is derived from the cubicovariant; a coincidence among 1′, 2′, 3′ implies a coincidence among 1, 2, 3, shown algebraically by the fact that the discriminant of the cubicovariant is a power of the discriminant of the cubic; the symmetric relation between the two triads of points proved in § 179 shows that the cubicovariant of the cubicovariant is not different from the original cubic, and it is seen that it is the original cubic multiplied by the square of the discriminant; and the fact that the points x_1, x_2 are the double points of the involution (11′, 22′, 33′) shows that the same two points will be found if we start from the cubicovariant.

296. As regards the quartic, $(a, b, c, d, e \between x, 1)^4$, we know three invariants (§§ 294, 155):—

(i.) the discriminant Δ;

(ii.) the expression g_2, $=ae-4bd+3c^2$, whose vanishing expresses that the roots are equianharmonic;

(iii.) the expression g_3, $=ace+2bcd-ad^2-b^2e-c^3$, whose vanishing expresses that the roots are harmonic; but these three are not independent. For the group of cross-ratios is determined by the equation (§ 156)

$$g_2^3\{(\phi+1)(\phi-2)(\phi-\tfrac{1}{2})\}^2=27g_3^2\{(\phi+\omega)(\phi+\omega^2)\}^3;$$

now a coincidence among the four points makes two of the cross-ratios equal to 0, and two equal to ∞; and as the sextic equation is

$$(g_2^3-27g_3^2)(\phi^6+1)-3(g_2^3-27g_3^2)(\phi^5+\phi)+A(\phi^4+\phi^2)+B\phi^3=0,$$

the necessary and sufficient condition for a coincidence of points is

$$g_2^3-27g_3^2=0.$$

Hence $g_2^3-27g_3^2$ is a numerical multiple of a power of Δ. This expression is of degree 6; and Δ, obtained by the elimination of x from

$$ax^4+4bx^3+6cx^2+4dx+e=0,$$
$$ax^3+3bx^2+3cx\ +d=0,$$

that is, from

$$ax^3+3bx^2+3cx\ +d=0,$$
$$bx^3+3cx^2+3dx+e\ =0,$$

is also of degree 6. Now as the term a^3e^3 occurs in $g_2^3-27g_3^2$ with coefficient unity, there is no numerical factor to be rejected from this expression; we find therefore

$$\Delta=g_2^3-27g_3^2,$$

and Δ does not count as a distinct invariant.

Using any two of the three invariants Δ, g_2^3, g_3^2, which are all of degree 6, the quotient is an absolute invariant; but of course one only of the six here indicated must be counted. Writing, for example, J for the quotient of g_2^3 by Δ, we have

$$J : J-1 : 1 = g_2^3 : 27g_3^2 : \Delta;$$

and the cross-ratio sextic can be written in any one of the forms

$$J : J-1 : 1 = 4(\phi^2-\phi+1)^3 : (2\phi^3-3\phi^2-3\phi+2)^2 : 27\phi^2(\phi-1)^2.$$

Note. Any one of the six cross-ratios determined by the four points is an invariant; but not being expressible rationally in terms of the coefficients, it is an irrational invariant; and the invariants g_2, g_3, Δ are symmetric functions of these six irrational invariants. Let the roots of the quartic be 1, 2, 3, 4, and write

$$l=(1-2)(3-4), \quad m=(1-3)(4-2), \quad n=(1-4)(2-3),$$

so that $$l+m+n=0;$$

then the six irrational invariants are the fractions

$$-\frac{m}{l}, \quad -\frac{n}{l}, \quad -\frac{n}{m}, \quad -\frac{l}{m}, \quad -\frac{l}{n}, \quad -\frac{m}{n}.$$

The expressions for g_2, g_3, Δ are

$$g_2=\frac{a^2}{24}(l^2+m^2+n^2)=-\frac{a^2}{12}(mn+nl+lm),$$

$$g_3=\frac{a^3}{432}(m-n)(n-l)(l-m),$$

$$\Delta=\frac{a^6}{256}l^2m^2n^2.$$

See Cayley, A Fifth Memoir upon Quantics, 1858; (*Collected Papers*, vol. ii., No. 156); and Klein, *Theorie der Elliptischen Modulfunctionen*, t. i., pp. 3-15. In Professor Cayley's Memoir, the invariants g_2, g_3 are denoted by I, J.

The quartic certainly has covariants; for calling the points 1, 2, 3, 4, we can associate these in different ways, so obtaining three involutions

$$(12,\ 34), \quad (13,\ 42), \quad (14,\ 23);$$

the three pairs of double points are symmetrically derived from 1, 2, 3, 4, hence the sextic equation by which they are determined has for its coefficients rational homogeneous expressions in a, b, c, d, e; they are projectively connected with 1, 2, 3, 4, hence the sextic is a covariant; and considering the mode of formation it is plain that the six points are ordinarily distinct, hence this is not a power of any lower covariant. The quartic has one more covariant; but no special purpose would be served by attempting a complete enumeration in this way; the examples given suffice to show how invariants and covariants present themselves in connection with a single binary quantic. In detecting invariants

or covariants by this process, two things must be attended to; in order that the derived function may be rationally expressible in terms of the coefficients of the given quantic, all the points must be involved symmetrically; and in order that the function may be endowed with the property of invariance, all the relations used must be projective.

297. As an example of invariants and covariants of a system, consider two quadratics,

$$ax^2+2bx+c=0, \quad a'x^2+2b'x+c'=0.$$

Here we have two pairs of points to consider; hence there is an invariant, $ac'+a'c-2bb'$, whose vanishing expresses that the two pairs are harmonic. And as the two pairs determine an involution, whose double elements are projectively related to the system, there is a covariant,

$$(ab'-a'b)x^2+(ac'-a'c)x+(bc'-b'c),$$

which equated to zero gives the double elements of the involution (§ 174).

Similarly with regard to a system of three quadratics there is an invariant,

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix},$$

whose vanishing expresses that the three pairs of points are in involution (§ 174).

Ex. Show by geometrical considerations that if an n-ic have a quadri-covariant, it has an n-ic covariant, which is not a product of lower covariants.

Ternary Quantics.

298. In dealing with ternary quantics, whose interpretation is by curves in a plane, we have to consider *invariants*—expressions whose vanishing indicates some permanent property of one curve, or relation of a system of curves; *covariants*—expressions which equated to zero give loci, having some permanent relation to the loci of the system (this system being supposed given in point coordinates, for definiteness); and also *contravariants*—expressions involving line coordinates, which equated to zero give envelopes having some permanent relation to the given system. For example, the envelope of a line cut harmonically by two conics S_1, S_2, is a conic Φ; this conic is a curve having a projective relation to the given pair of conics; if then we express its equation in terms of x, y, z, we have a covariant of the system of two

conics. But its equation presents itself most naturally in line coordinates ξ, η, ζ. In this form it is not an invariant of the given system, for it is not expressed in terms of the coefficients only; it is not a covariant, for it does not involve the variables x, y, z; nevertheless it is endowed with the property of invariance, and it is called a contravariant (French, *Forme adjointe*; German, *Zugehörige Form*). Reciprocally, if a system be given in line coordinates, invariant expressions involving ξ, η, ζ are covariants, and invariant expressions involving x, y, z are contravariants.

The general term invariant, explicitly referring to the permanence of relation, is often used as including covariants and contravariants. In this general sense, an invariant of an invariant is an invariant; in the special sense, an invariant of a covariant or contravariant is an invariant; a covariant of a covariant, or a contravariant of a contravariant, is a covariant; and a covariant of a contravariant, or a contravariant of a covariant, is a contravariant.

299. The idea of two distinct sets of coordinates, to which we are led in analytical geometry by the principle of duality, is deliberately accepted and generalized in the algebra of linear transformations; that is, as stated in § 285, the group of collineations is extended by the addition of linear dualistic transformations. Reference to § 34 shows that the relation of the two sets of coordinates exhibits itself in linear transformation in the fact that they are transformed by inverse substitutions; that is, the formulæ of transformation for the one set being

$$\begin{aligned} x &= l_1x' + m_1y' + \ldots, \\ y &= l_2x' + m_2y' + \ldots, \\ &\text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

those for the other set are

$$\begin{aligned} \xi' &= l_1\xi + l_2\eta + \ldots, \\ \eta' &= m_1\xi + m_2\eta + \ldots, \\ &\text{etc.}\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and the two sets are connected by the identical relation

$$x\xi + y\eta + z\zeta + \ldots = x'\xi' + y'\eta' + z'\zeta' + \ldots.$$

This is adopted as the defining property in algebra; sets of quantities that are transformed by the same substitution are called *cogredient*; sets that are transformed by inverse substitutions are *contragredient*. Hence sets of point coordinates are cogredient; sets of line coordinates are cogredient; but point and line coordinates are contragredient.

Thus modern algebra does not stop with the projective two-dimensional geometry we are now considering, it admits any number of variables; "its group consists of the totality of linear and dualistic transformations of the variables employed to represent individual configurations in the manifoldness; it is the generalization of projective geometry" (Klein).

300. Any contravariant expression can, however, be regarded in a different way. Taking the example already used, it was required to find the envelope of a line cut harmonically by two conics. But this may be stated in the form:—What condition must $l:m:n$ satisfy in order that the line $L=lx+my+nz=0$ may be cut harmonically by the conics $S_1=0$, $S_2=0$? This is a question as to the permanent relation of the line L and the conics S_1, S_2; the answer is that a certain invariant of the system S_1, S_2, L must vanish. Thus when we regard ξ, η, ζ as known, Φ is an *invariant* of the system S_1, S_2, L; when we regard ξ, η, ζ no longer as known quantities, but as line coordinates, Φ is a *contravariant* of the system S_1, S_2; and when we express the equation of the conic Φ in point coordinates, so obtaining V, V is a *covariant* of the system S_1, S_2. Hence there is no absolute necessity for considering contravariants in the theory of ternary quantics; a contravariant is simply the reciprocal form of a covariant; the reason for admitting the two conceptions is the same as the reason for using both point and line coordinates.

Any curve $S=0$ has a line equation $\Sigma=0$, that is, in dealing with ternary quantics, to every single expression there is certainly one contravariant. As regards binary quantics, contravariants have no special significance. The connecting equation for the two sets of coordinates being in this case $x\xi+y\eta=0$, we have $\xi:\eta=y:-x$, and nothing is to be gained by the use of ξ, η.

301. Finally, both sets of coordinates may enter into an invariant expression; it is then called a *mixed concomitant* (French, *Forme mixte*; German, *Zwischenform*). For example, the equation of the pair of tangents to a conic S at the points where it is met by a line L involves the coefficients in S, the coordinates x, y, z, and the coefficients l, m, n in L; if l, m, n be regarded as known, the result is a covariant of the system S, L; but if l, m, n be regarded as coordinates of the line, and written ξ, η, ζ, the result is a mixed concomitant.

302. When considering the properties of two or three

conics, it was found that some relate to all conics of the pencil or net. For example, if a line be cut in involution by three conics S_1, S_2, S_3, it is cut in involution by every conic of the net. Regarding the line L as known, the fact that it is cut in involution by the three conics, being projective, is expressed by the vanishing of a certain expression G; G is an invariant of the system S_1, S_2, S_3, L; and as regards the conics S_1, S_2, S_3, it is a *combinant*; and when the coefficients in L are regarded as line coordinates, so that G is a contravariant of S_1, S_2, S_3, it is still a *combinant*. "An invariant of a system of quantics of the same degree is a combinant if it be unaltered when for any of the quantics is substituted a linear function of the quantics" (Salmon).

Combinants present themselves in the theory of binary quantics. Consider two quadratics,

$$u=ax^2+2bx+c, \quad v=a'x^2+2b'x+c'.$$

These have a covariant (§ 297),

$$f=(ab'-a'b)x^2+(ac'-a'c)x+(bc'-b'c),$$

which equated to zero gives the double points of the involution determined by the two pairs $u=0$, $v=0$. But every pair $u+\lambda v=0$ is in this involution; hence the covariant f is a combinant. Similarly the three pairs $u=0$, $v=0$, $w=0$ are in involution if a certain invariant vanish (§ 297); and in this case all pairs $u+\lambda v+\mu w=0$ belong to the involution; the invariant is a combinant.

303. Considering a single conic, there is one invariant, Δ, whose vanishing expresses that the conic is degenerate; and there is a contravariant, the reciprocal equation of the conic. A proper conic by itself has no projective property; and there is no locus and no envelope projectively connected with it. To obtain the metric properties of a conic we have to consider the projective relations of two conics, and then make one degenerate into a pair of points. Hence the importance of the theory of the invariants of a system of two conics.

Supposing the conics to be given in point coordinates, a number of the special relations of position can be expressed in terms of the four common points A, B, C, D. Now these common points are most simply assigned by means of straight lines joining them in pairs; we require therefore the common chords of the conics, that is, the line-pairs included in the pencil $kS+S'=0$. The three values of k giving the three line-pairs are the roots of the cubic

$$\Delta k^3+\Theta k^2+\Theta' k+\Delta'=0 \quad \text{..................(1)},$$

where S, S' being written in the ordinary form, Δ, Δ' are the discriminants, and

$$\Theta = Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh',$$
$$\Theta' = A'a + B'b + C'c + 2F'f + 2G'g + 2H'h,$$

$A, B, \ldots A', B', \ldots$ being written for the minors of $a, b, \ldots a', b', \ldots$ in the determinants Δ, Δ'.

Note. The notation here used is that of Chap. XVIII. in Salmon's *Conic Sections*; this work will now be referred to as *C.*

If $kS + S'$ be degenerate, it is degenerate after linear transformation; hence the values of k given by equation (1) are unalterable by transformation; the coefficients in the equation are invariants; the vanishing of any one of these coefficients expresses some projective property of the system of two conics. The significance of $\Delta = 0$, $\Delta' = 0$ is known; for the meaning of the vanishing of Θ, Θ' respectively see *C.* § 375.

The conics have contact if two of their common points coincide; since the coincidence of A, B causes the line-pair AC, BD to coincide with BC, AD, the condition of contact is found by expressing that equation (1) has equal roots (*C.* § 372); and since every conic of the pencil passes through A, B, any two conics of the pencil have contact. The invariant whose vanishing expresses contact, that is, the *tact-invariant*, is therefore a combinant.

304. Since the special importance of the system of two conics consists in its affording a means of investigating the metric properties of one conic, the second conic of the system is to be degenerate, in line coordinates. If S' be degenerate, Δ' vanishes, and equation (1) reduces to the quadratic

$$\Delta k^2 + \Theta k + \Theta' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Supposing the equations to be in point coordinates, $S' = 0$ is a line-pair, AC, BD. The other line-pairs are AB, CD and AD, BC. If A, B coincide, the pair AD, BC is the same as S', and the intersection of the two lines S' is the point A (or B), that is, a point on S. Now for the pair AD, BC to be the same as S', the corresponding value of k must be zero, hence $\Theta' = 0$. Thus if S' be a line-pair, and $\Theta' = 0$, the intersection of the lines S' is a point on S (*C.* § 373). If however $\Theta = 0$, the lines S' are conjugate with respect to S (*C.* § 373).

Reciprocally, if the equations be in line coordinates, S' being degenerate is a point-pair; $\Theta' = 0$ is the condition that the conic S touch the line joining the points S'; and $\Theta = 0$ is the condition that the points S' be conjugate with respect

to the conic S. Thus if $S'=0$ represent the circular points, the condition that the conic be a parabola is $\Theta'=0$, and the condition for a rectangular hyperbola is $\Theta=0$, these conditions being formed from the line equation of the conic.

Ex. Deduce the ordinary Cartesian conditions for a parabola and a rectangular hyperbola.

305. A full discussion of the invariants of a system of two conics is contained in Chapter XVIII. of Salmon's *Conic Sections*. Reference may also be made with advantage to Chapter XV. of Casey's *Analytical Geometry* (edition of 1893), where, on p. 517, there will be found a complete list of the concomitants of two conics; and to Clebsch, *Vorlesungen über Geometrie*, t. i., pp. 265-304. These books being well known there is no occasion to repeat the discussion here; the object of this chapter is simply to show that the language of the algebra of linear transformations does exactly express the projective geometry to which the preceding chapters are devoted.

INDEX.

(The numbers refer to the pages; when a number is printed in italics, the reference is to an example on the page indicated.)

END.

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